

A MODEL FOR NON-FREE CIRCLE ACTIONS

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Abstract

Let $\Phi: \mathbb{S}^1 \times M \rightarrow M$ be a smooth action of the unit circle \mathbb{S}^1 on a manifold M . In this work, we compute the minimal model of M in terms of the orbit space B and the fixed point set $F \subset B$, as a dg-module over the Sullivan's minimal model of B .

The question we treat in this work is the following, given a smooth action $\Phi: \mathbb{S}^1 \times M \rightarrow M$, it is possible to construct a model of M using just basic data? The answer is wellknown when the fixed point set is empty (in particular, when the action is free). A dgca model of M is given by a Hirsch extension of the dgca Sullivan minimal model $\mathcal{A}(B)$ of the orbit space B

$$(1) \quad \mathcal{A}(B) \otimes \Lambda(x),$$

where $\deg x = 1$ and dx defines the Euler form of the action (see for example [6]). This formula does not apply when the fixed point set F is not empty. Roughly speaking this happens because the Euler form does not live on B .

Our answer to the above question is a minimal model of the deRham dgc algebra $\Omega(M)$ of M , which is a dg module over the dgc Sullivan minimal algebra $\mathcal{A}(B)$. Such structure is associated to M by means of the canonical projection $\pi: M \rightarrow B$. We prove that the minimal model¹ of M is the graded cone

$$(2) \quad \mathcal{M}(M) = \mathcal{A}(B) \oplus_{e'} \mathcal{M}(B, F),$$

where $e': \mathcal{M}(B, F) \rightarrow \mathcal{A}(B)$ is a map of degree 2. This map is determinated by the Euler class of the action and it will be described below. Notice that this formula generalizes (1) since, for $F = \emptyset$, formulae (1) and (2) become $\mathcal{A}(B) \oplus_e \mathcal{A}(B)$.

There are some algebraic invariants of M and F that are closely related: Poincaré characteristic, localization, rational homotopy, ... We add another item to this list: the minimal model of M and F . In fact, considering the $\mathcal{A}(B)$ -dg module structure associated to F by means of the natural inclusion $\iota: F \hookrightarrow B$, we prove that the minimal model of F is the graded cone

$$\mathcal{M}(F) = \mathcal{A}(B) \oplus_{i'} \mathcal{M}(B, F),$$

where $i': \mathcal{M}(B, F) \rightarrow \mathcal{A}(B)$ is a map of degree 0. This map is determinated by ι and it will be described below. Observe that the minimal models $\mathcal{M}(M)$ and $\mathcal{M}(F)$ have, as $\mathcal{A}(B)$ -graded modules, the same basis up to a shift by 2.

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¹In this category minimal models will be denoted by $\mathcal{M}(-)$.

We are also interested on the Borel space $M_{\mathbb{S}^1} = M \times_{\mathbb{S}^1} \mathbb{S}^\infty$, where an $\mathcal{A}(B)$ -dg module structure is associated by means of the canonical projection $p: M_{\mathbb{S}^1}^1 \rightarrow B$. We prove that the minimal model of $M_{\mathbb{S}^1}^1$ is the graded cone

$$\mathcal{M}(M_{\mathbb{S}^1}) = [\mathcal{A}(B) \otimes \Lambda(e)] \oplus_{q'} [\mathcal{M}(B, F) \otimes \Lambda(e)],$$

where $\deg e = 2$, $de = 0$ and $q'(b \otimes e^n) = e'(b) \otimes e^n + i'(b) \otimes e^{n+1}$. This formula implies that the equivariant cohomology $H_{\mathbb{S}^1}^i(M)$ (i.e. the cohomology of $M_{\mathbb{S}^1}$) can be computed using just basic data by means of the long exact sequence

$$\cdots \rightarrow [H(B) \otimes \Lambda(e)]^i \rightarrow H_{\mathbb{S}^1}^i(M) \rightarrow [H(B, F) \otimes \Lambda(e)]^{i-1} \xrightarrow{(q')^*} [H(B) \otimes \Lambda(e)]^{i+1} \rightarrow \cdots.$$

Moreover, when the Euler class vanishes, we prove that the equivariant cohomology of M is just $H(B) \oplus [H(F) \otimes \Lambda^+(e)]$. We also translate some classic results (Localization Theorem, equivariantly formality, ...) in terms of basic data.

Let us illustrate these results with the suspension of the Hopf action on \mathbb{S}^3 . The north and south poles of the total space \mathbb{S}^4 are the fixed points of the action and the orbit space is \mathbb{S}^3 . We work in the category of $\Lambda(a)$ -dg modules, where $\deg a = 3$. From the above formulae we get:

$$\begin{aligned} \mathcal{M}(\mathbb{S}^4) &= \Lambda(a) \otimes \mathbb{R}\{1, c_n \mid n \in \mathbb{N}\}, \text{ with} \\ \deg c_n &= 2\left[\frac{n+3}{2}\right], dc_0 = a, dc_1 = 0, dc_{n+2} = a \cdot c_n. \end{aligned}$$

$$\begin{aligned} \mathcal{M}(\mathbb{S}^0) &= \Lambda(a) \otimes \mathbb{R}\{1, \gamma_n \mid n \in \mathbb{N}\}, \text{ with} \\ \deg \gamma_n &= 2\left[\frac{n+1}{2}\right], d\gamma_0 = 0, d\gamma_1 = a, d\gamma_{n+2} = a \cdot \gamma_n. \end{aligned}$$

$$\begin{aligned} \mathcal{M}(\mathbb{S}^4 \otimes_{\mathbb{S}^1} \mathbb{S}_\infty) &= \Lambda(e, a) \otimes \mathbb{R}\{1, c_n \mid n \in \mathbb{N}\}, \text{ with} \\ \deg c_n &= 2\left[\frac{n+3}{2}\right], dc_0 = a, dc_1 = e \cdot a, dc_{n+2} = a \cdot c_n. \end{aligned}$$

The minimal model $\mathcal{M}(\mathbb{S}^4)$ (resp. $\mathcal{M}(\mathbb{S}^0)$, resp. $\mathcal{M}(\mathbb{S}^4 \otimes_{\mathbb{S}^1} \mathbb{S}_\infty)$) is a free $\Lambda(a)$ -graded module over the cohomology $H^*(Y_\pi)$ of the homotopy fiber Y_π of π (resp. Y_ι , resp. Y_p). So, we find the following relations between the Poincaré polynomials of these spaces:

$$P_{Y_\pi} = 1 - t^2 + t^2 P_{Y_\iota} = (1 - t^2) P_{Y_p}.$$

We prove that these relations are generic if B is simply connected and of finite type.

The main geometric tool used in this work is the notion of controlled form of Verona [21]. In fact, when the set of fixed points F is not empty the orbit space B is not a regular manifold but a singular one, more precisely a stratified pseudomanifold. For a such space Z , Verona proved that the complex of controlled forms $\Omega_v^*(Z)$ compute the cohomology of Z . We prove more, the minimal models $\mathcal{A}(Z)$ and $\mathcal{M}(Z)$ can be computed using controlled forms. It is important to notice that the Euler form is not a controlled form, nevertheless it appears in this context as a morphism of $\mathcal{A}(B)$ -dg modules $e: \Omega_v^*(B, F) \rightarrow \Omega_v^{*+2}(B)$ (cf. [10]). In the writing of $\mathcal{M}(M)$ (resp. $\mathcal{M}(F)$) the operator e' (resp. i') is a model of e (resp. of the inclusion $i: \Omega_v^*(B, F) \hookrightarrow \Omega_v^*(B)$).

The starting point of the work is the observation that the cohomology of M can be computed by the graded cone $\Omega_v^*(B) \oplus_e \Omega_v^*(B, F)$. This formula also applies to semi-free actions of \mathbb{S}^3 [20]. So, all the results of this work extend to this kind of actions. A similar formula appears when one deals with an isometric action $\Phi: \mathbb{R} \times M \rightarrow M$, considering on B controlled basic forms instead

controlled forms [20]. Again, we conclude that the results of this work apply to isometric flows. In particular, we get the inequality

$$H^{r-1}((M, F)/\mathcal{F}) + \sum_{i=0}^{\infty} \dim H^{r+2i}(F) \leq \sum_{i=0}^{\infty} \dim H^{r+2i}(M),$$

when the flow is not trivial.

On the algebraic side, we develop in some extend the Theory of dg minimal modules. This kind of minimal objects was previously studied by the first author (cf. [15], [16], [17]) and independently by Gugenheim and May (cf. [7]).

The organization of the work is as follows. First section is devoted to present the algebraic tools we need to work with A -dg modules. In the second section we present the singular spaces we find when we deal with circle actions. Controlled forms are introduced in the third section. The main result of this paper is proved in the forth and last section. Four technical Lemma are proven in the Appendix.

A manifold is considered to be connected, without boundary and smooth (of class c^∞), unless otherwise is stated. The field of coefficients is \mathbb{R} .

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1 Dg-Module minimal models

In this section we will develop the algebraic machinery necessary to prove Theorem 4.3.1: we define what is meant by a *minimal factorization of a morphism* of A -dg modules, prove its existence and uniqueness and a result concerning maps induced between them.

1.1 A -dg modules. Let A be a dgc algebra. An A -dg module M is a graded vector space together with a product $A \otimes M \rightarrow M$ and a differential $d: M \rightarrow M$ of degree $+1$ which satisfies Leibniz rule. Both the graduation of A and M are over the non negative integers. A *quasi-isomorphism* (*quis*) is an A -dg module morphism which induces an isomorphism in cohomology.

Let us begin with an immediate generalization of the cone of a morphism of complexes in the category of A -dg modules. Let $\varphi: M \rightarrow N$ be a morphism of A -dg modules of degree $p \in \mathbb{Z}$. This is the same as a degree 0 morphism of A -dg modules

$$\varphi: M[-p] \rightarrow N$$

where $M[-p]$ means the A -dg module M shifted by $-p$; it is graded by $M[-p]^n = M^{n-p}$ and the product $\mu_{M[-p]}: A \otimes M[-p] \rightarrow M[-p]$ and the differential $d_{M[-p]}: M[-p] \rightarrow M[-p]$ change signs according to

$$\begin{aligned} \mu_{M[-p]}(a \otimes m) &= (-1)^{|a| \cdot p} \mu_M(a \otimes m) \\ d_{M[-p]}(m) &= (-1)^p d_M(m). \end{aligned}$$

We shall denote by $N \oplus_\varphi M$ the A -dg module graded by

$$N \oplus_\varphi M = N \oplus M[1-p]$$

and with product and differential given by the formulas:

$$a \cdot \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} a \cdot y \\ (-1)^{|a|(p-1)} a \cdot x \end{pmatrix}$$

$$\begin{pmatrix} d & \varphi \\ 0 & (-1)^{p-1}d \end{pmatrix} \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} dy + \varphi x \\ (-1)^{p-1}dx \end{pmatrix},$$

where $a \cdot x$ denotes $\mu_M(a \otimes x)$. Finally, when we say that the sequence of A -dg module morphisms

$$0 \longrightarrow M \xrightarrow{\varphi} N \xrightarrow{\psi} P \longrightarrow 0$$

is *exact*, we simply mean that $0 \longrightarrow M^n \xrightarrow{\varphi^n} N^n \xrightarrow{\psi^n} P^n \longrightarrow 0$ are usual exact sequences of A^0 -modules for all n .

1.1.1 Remark. A short exact sequence of A -dg modules like above is the same as a quis of A -dg modules

$$N \oplus_{\varphi} M \xrightarrow{(\psi, 0)} P.$$

Lemma 1.1.2 *Given a morphism $\varphi: M \rightarrow N$ of A -dg modules of degree p , we have a short exact sequence of A -dg modules*

$$0 \longrightarrow N \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} N \oplus_{\varphi} M \xrightarrow{(0, 1)} M[1-p] \longrightarrow 0.$$

Proof. Obvious. ♣

Associated to the above exact sequence, there is the *long exact cohomology exact sequence* of φ :

$$\dots \rightarrow H^n(N) \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}_*} H^n(N \oplus_{\varphi} M) \xrightarrow{(0, 1)_*} H^{n+1-p}(M) \xrightarrow{\varphi_*} H^{n+1}(N) \rightarrow \dots$$

To end this elementary differential homological algebra, let us point out that a (*degree p*) *homotopy* between two A -dg module morphisms $\varphi, \psi: M \rightarrow N$ of degree p , is an A -dg module morphism $h: M \rightarrow N$ of degree $p-1$, such that

$$(-1)^p dh + hd = \psi - \varphi.$$

One can verify that this notion of homotopy coincides with the one defined in [17] using a path object.

1.2 Minimal A -dg modules. (cf. [8], [12], [15]) Let M be a A -dg module and n a non negative integer. A *degree n Hirsch extension* of M is an inclusion of A -dg modules $M \hookrightarrow M \oplus_d (A \otimes V^n)$ in which:

1. V is a homogeneous vector space of degree n ,
2. $A \otimes V^n$ is the free A -graded module over V , and
3. $d: V^n \rightarrow M^{n+1}$ is a linear map.

A morphism of A -dg modules $M \oplus_d (A \otimes V^n) \longrightarrow N$ is given by a morphism of A -dg modules $\varphi: M \longrightarrow N$ and a linear map $f: V^n \longrightarrow N^n$ subjected to the condition $\varphi \circ d = d \circ f$.

A *minimal KS-extension* of M is an inclusion of A -dg modules $\iota: M \longrightarrow N$ together with an exhaustive filtration $\{N(n, q)\}_{(n, q) \in I}$ of N , indexed by $I = \{(n, q) \in \mathbb{N} \times \mathbb{N}\}$ with lexicographical order, such that:

1. $N(0, 0) = M$,
2. for $q > 0$, $N(n, q)$ is a degree n Hirsch extension of $N(n, q - 1)$, and
3. $N(n + 1, 0) = \varinjlim_q N(n, q)$.

1.2.1 Remark. So N is of the form $M \oplus_d (A \otimes V)$, where V is a bigraded vector space. This kind of objects plays the rôle of the KS-extensions of [8], noted $B \otimes \Lambda V$, with the tensor product replaced by the direct sum and the free dg-algebra over V replaced by the free A -dg module over V .

A *minimal KS-factorization* of an A -dg module morphism $\varphi: M \longrightarrow N$ is a commutative diagram of A -dg module morphisms

$$(3) \quad \begin{array}{ccc} & M & \\ \iota \swarrow & & \searrow \varphi \\ M \oplus_d (A \otimes V) & \xrightarrow{\rho} & N \end{array}$$

in which ρ is a *quis* and ι is a minimal KS-extension. If M is the zero A -dg module, we talk about *minimal KS-modules* and *minimal KS-models*.

1.3 Models of A -dg modules. Let now (3) be any commutative diagram of A -dg module morphisms. In this situation, we will say that ρ is a *M -morphism*. If ρ is also a *quis*, we will simply say that it is a *M -quis*. A homotopy between two M -morphisms which restricted to M is the identity will be called also a *M -homotopy*.

Alternatively, we could have said that ρ is a morphism of $M \backslash \mathbf{DGM}(A)$, the category of A -dg modules under M . So, a minimal KS-factorization is, simply, a minimal model in $M \backslash \mathbf{DGM}(A)$ (see [17] for the precise statement of this).

Theorem 1.3.1 *Let A be a dgc algebra and let $\varphi: M \longrightarrow N$ be an A -dg module morphism such that $\varphi_*^0: H^0(M) \longrightarrow H^0(N)$ is a monomorphism. Then there exists a minimal KS-factorization of φ .*

Proof. See Appendix. ♣

Corollary 1.3.2 *Let A be a dgc algebra and let N be an A -dg module. Then there exists a minimal KS-model of N .*

Theorem 1.3.3 (cf. [9], [15]) *Let A be a dgc algebra and*

$$\begin{array}{ccc} & M & \\ \psi \swarrow & & \searrow \iota \\ X & \xrightarrow{\varphi} & M \oplus_d (A \otimes V) \end{array}$$

a commutative diagram of A -dg module morphisms in which ι is a minimal KS-extension and φ is a quis. Then there exists an A -dg module morphism $\sigma: M \oplus_d (A \otimes V) \rightarrow X$ such that $\sigma\iota = \psi$ and $\varphi\sigma = \text{id}$.

Proof. See Appendix. ♣

In other words, every M -quis whose target is a minimal KS-extension of M has a section which also is a M -morphism. This implies, the well-known, in other categories, uniqueness up to isomorphism of minimal models.

Corollary 1.3.4 *Two minimal KS-factorizations of the same M -morphism are M -isomorphic and the isomorphism is unique up to M -homotopies.*

Proof. It follows easily from Theorem 1.3.3, taking into account that $M \backslash \mathbf{DGM}(A)$ is a closed model category in which all objects are fibrant (cf. [14], [17, Corollary 2 to Proposition 1.15]). ♣

Particularly, if we take $M = 0$, the zero A -dg module, we obtain

Corollary 1.3.5 *Two minimal KS-models of the same A -dg module are isomorphic and the isomorphism is unique up to homotopy.*

Given a morphism $\varphi: M \rightarrow N$ of A -dg modules we will need to construct a model of $N \oplus_\varphi M$. This will be done by means of the following results:

Corollary 1.3.6 *Let $\varphi: M \rightarrow N$ be a morphism of A -dg modules and let $\rho_M: M' \rightarrow M$ and $\rho_N: N' \rightarrow N$ be two minimal models. Then there exists a morphism $\varphi': M' \rightarrow N'$, unique up to homotopies, that renders commutative up to homotopy the diagram*

$$\begin{array}{ccc} M' & \xrightarrow{\rho_M} & M \\ \downarrow \varphi' & & \downarrow \varphi \\ N' & \xrightarrow{\rho_N} & N \end{array}$$

We will loosely say that φ' is a model of φ .

Proof. It follows from Theorem 1.3.3 and the model category structure [16, Proposition 1.16]. ♣

Proposition 1.3.7 *Consider the above diagram. Let $h: M' \rightarrow N$ be a homotopy between $\varphi \circ \rho_M$ and $\rho_N \circ \varphi'$. Then*

$$\Phi = \begin{pmatrix} \rho_N & h \\ 0 & \rho_M \end{pmatrix} : N' \oplus'_\varphi M' \rightarrow N \oplus_\varphi M$$

is a quis.

Proof. Let us verify that Φ commutes with differentials.

$$\begin{pmatrix} d & \varphi \\ 0 & (-1)^{p-1}d \end{pmatrix} \begin{pmatrix} \rho_N & h \\ 0 & \rho_M \end{pmatrix} - \begin{pmatrix} \rho_N & h \\ 0 & \rho_M \end{pmatrix} \begin{pmatrix} d & \varphi \\ 0 & (-1)^{p-1}d \end{pmatrix} = \\ \begin{pmatrix} d\rho_N - \rho'_N & dh + (-1)^p h d - (\rho_N \varphi' - \varphi \rho_M) \\ 0 & (-1)^{p-1} d\rho_M - (-1)^{p-1} \rho_M d' \end{pmatrix}.$$

And this is zero because ρ_M and ρ_N commute with differentials and h is a degree p homotopy between $\varphi \circ \rho_M$ and $\rho_N \circ \varphi'$. Next, we put Φ in the following obviously commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & N & \longrightarrow & N \oplus_{\varphi} M & \longrightarrow & M[1-p] \rightarrow 0 \\ & & \uparrow \rho_N & & \uparrow \Phi & & \uparrow \rho_M \\ 0 & \longrightarrow & N' & \longrightarrow & N' \oplus'_{\varphi'} M' & \longrightarrow & M'[1-p] \rightarrow 0 \end{array}$$

And so Φ is a quasis by the long exact sequence of φ and φ' and the Five Lemma. ♣

Finally, we will need the following result concerning minimal dg modules and graded cones.

Lemma 1.3.8 *Let $\varphi: M \rightarrow A$ be a degree p morphism of A -dg modules, with M a minimal A -dg module. If $M^{<(1-p)} = 0$ then $A \oplus_{\varphi} M$ is a minimal A -dg module.*

Proof.

Let $M = A \otimes V$. Then, as a graded module, $A \oplus M = A \otimes (\mathbb{R} \oplus V)$. So we can define an exhaustive filtration in $A \oplus M$ as follows. Let $W(n, q)$ be the \mathbb{R} -vector spaces

$$W(n, q) = \begin{cases} 0 & \text{if } n = q = 0 \\ \mathbb{R} & \text{if } n = 0 \text{ and } q = 1 \\ V(n+1-p, q-1) & \text{if } n = 0 \text{ and } q > 1 \\ V(n+1-p, q) & \text{if } n \neq 0 \end{cases}$$

and put

$$(A \oplus_{\varphi} M)(n, q) = A \otimes \left(\bigoplus_{(m, r) \leq (n, q)} W(m, r) \right).$$

Then, all the inclusions $(A \oplus_{\varphi} M)(n, q-1) \hookrightarrow (A \oplus_{\varphi} M)(n, q)$ are degree n Hirsch extensions. ♣

1.3.9 Links with Topology. Before studying the relative case, let us show one example where these *minimal dg-modules* appear in topology. Let $p: E \rightarrow B$ be a continuous map between two topological spaces. We have the induced morphism $p^*: A_{\mathbb{R}}(B) \rightarrow A_{\mathbb{R}}(E)$ between the real algebras of polynomial forms making $A_{\mathbb{R}}(E)$ an $A_{\mathbb{R}}(B)$ -dg module. Now, assume that B is connected, of finite type with $\pi_1(B)$ acting trivially on $H^*(Y_p) = H^*(Y_p, \mathbb{R})$, where Y_p denotes the *homotopy fiber* of p . Then, by the second Theorem of Eilenberg-Moore (see [7] and [1]), we have that

$$H^*(Y_p) \cong \text{Tor}_{A_{\mathbb{R}}(B)}(\mathbb{R}, A_{\mathbb{R}}(E)).$$

By [17], this differential torsion product can be computed with a minimal model of $A_{\mathbb{R}}(E)$ as an $A_{\mathbb{R}}(B)$ -dg module. Let $\mathcal{M}(E) = A_{\mathbb{R}}(B) \otimes_{\mathbb{R}} V$ be this minimal model. Then

$$H^*(Y_p) \cong H^*(\mathbb{R} \otimes_{A_{\mathbb{R}}(B)} \mathcal{M}(E)) \cong H^*(\mathbb{R} \otimes_{A_{\mathbb{R}}(B)} (A_{\mathbb{R}}(B) \otimes V)) = V$$

because $\mathbb{R} \otimes_{A_{\mathbb{R}}(B)} (A_{\mathbb{R}}(B) \otimes V) = V$ has zero differential due to minimality. So, the known model of Hirsch-Brown of E

$$\mathcal{M}(E) = A_{\mathbb{R}}(B) \otimes H^*(Y_p)$$

is a *minimal model* as $A_{\mathbb{R}}(B)$ -dg modules. In particular, if we take B to be a point, we find that $H^*(E)$ is the minimal model of $A_{\mathbb{R}}(E)$ as a \mathbb{R} -dg module.

1.4 Models of couples, existence and uniqueness. In fact, we will need something more than simply dg-minimal models over a fixed \mathbb{R} -dg algebra. The process we are going to perform is the following: starting with an A -dg module M , we are going to compute first the dgca minimal model of A :

$$\rho: A' \xrightarrow{\sim} A$$

(i.e., the classical Sullivan minimal model). Then, by means of the dgca quis ρ , we will make M an A' -dg module by defining the product with elements of A' by

$$a \cdot m = \rho^*(a) \cdot m,$$

where the product in the right-hand side is the product of M as an A -module. Let us note $\rho^*(M)$ the dg module M with this structure of A' -module. Finally, we will compute the minimal model of $\rho^*(M)$ as a A' -dg-module:

$$\varphi: M' \xrightarrow{\sim} \rho^*(M).$$

In other words, we will compute the minimal model of the couple (A, M) in the category **DGM** of *modules over all algebras*. The objects of this category are couples like (A, M) . Morphisms are also couples

$$(f, \varphi): (A, M) \rightarrow (B, N)$$

where $f: A \rightarrow B$ is a morphism of dgc algebras and $\varphi: M \rightarrow N$ is an *dg-module f -morphism*; that is to say, a morphism of A -dg modules $M \rightarrow f^*(N)$. In other words:

$$\varphi(a \cdot m) = f(a) \cdot \varphi(m) \quad \text{for all } a \in A, m \in M.$$

The algorithm we have described brings us the “true” minimal model

$$(g, \psi): (A', M') \rightarrow (A, M)$$

in the sense that the couple (A', M') is unique up to isomorphism (of **DGM**) and the couple of quis (g, ψ) is also unique up to homotopies (of **DGM**); this follows from [17, Theorem 3.4], which tells us that the couple (A, M) is minimal (in the sense that is unique up to isomorphism) in **DGM** if and only if A is a minimal \mathbb{R} -dgc algebra and M is an A -dg minimal module.

2 Stratifications and unfoldings

We fix in this work a smooth action $\Phi: \mathbb{S}^1 \times M \rightarrow M$ (no trivial!). The orbit space of the action is B and $\pi: M \rightarrow B$ is the canonical projection. The action Φ induces on M a natural stratification by classifying the points of M according to their isotropy subgroups. This stratification is invariant by the action of \mathbb{S}^1 , so the orbit space B inherits also a stratified structure. In this section we precise these facts.

2.1 Stratifications. A *stratification* of a paracompact topological space Z is a locally finite collection \mathcal{S}_Z of disjoint manifolds called *strata*, such that

- i) $Z = \bigsqcup_{S \in \mathcal{S}_Z} S$.
- ii) $S \cap \overline{S'} \neq \emptyset \iff S \subset \overline{S'}$ (and we write $S \leq S'$).
- iii) (\mathcal{S}_Z, \leq) is a partially ordered set (*poset*).
- iv) There exists an open stratum R which is the maximum.

We shall say that Z is a *stratified space*. Notice that R , called *regular stratum*, is necessarily dense. A *singular stratum* is an element of \mathcal{S}_Z different from R . We shall write \mathcal{S}_Z^{sing} the family of singular strata and $\Sigma_Z \subset Z$ its union. The *length* of Z , written $\text{len } Z$ is the biggest integer n for whom there exists a chain $S_0 < S_1 < \dots < S_n$ of strata. In particular, $\text{len } Z = 0$ if and only if Z is manifold endowed with the stratification $\mathcal{S}_Z = \{\text{connected components of } Z\}$. Notice that the length is always finite.

A continuous map (resp. homeomorphism) $f: Y \rightarrow Z$ between two stratified spaces is a morphism (resp. *isomorphism*) if it sends smoothly (resp. diffeomorphically) the strata of Y to the strata of Z . We shall write $\text{Iso}(Z)$ the group of isomorphisms between Z and itself. A morphism $f: Y \rightarrow Z$ induces a poset morphism $f_S: \mathcal{S}_Y \rightarrow \mathcal{S}_Z$ by putting $f_S(S) \supset f(S)$. We shall say that f is a *strict morphism* if the map f_S is strictly increasing.

2.1.1 Examples. Through this work we shall find the following kinds of stratification.

- (a) On a connected manifold N we always may consider the 0-length stratification $\mathcal{S}_N = \{N\}$. A stratum $S \subset Z$ inherits from \mathcal{S}_Z a such stratification.
- (b) Any open subset W of Z inherits naturally from \mathcal{S}_Z a stratified structure satisfying $\text{len } W \leq \text{len } Z$. The stratification is $\mathcal{S}_W = \{\text{connected components of } S \cap W \mid S \in \mathcal{S}_Z\}$. Notice that the inclusion $W \hookrightarrow Z$ is a strict morphism.
- (c) Suppose Z compact. On the product $N \times cZ$, where cZ is the *cone* $Z \times [0, 1[/ Z \times \{0\}$, we have the stratification $\mathcal{S}_{N \times cZ} = \{N \times S \times]0, 1[\mid S \in \mathcal{S}_Z\} \cup \{N \times \{\text{vertex } \vartheta \text{ of } cZ\}\}$. Notice that $\text{len } N \times cZ = \text{len } Z + 1$. A point of cZ will be denote by $\llbracket x, t \rrbracket$ with $(x, t) \in Z \times [0, 1]$. The vertex ϑ of cZ is $\llbracket x, 0 \rrbracket$.

Unless otherwise stated, we assume that the spaces W , cZ and $N \times cZ$ are endowed with the stratification described above. Later on, we shall show how Φ determines a natural stratification on M and B .

2.2 Stratified pseudomanifolds. When the strata are assembled in a conical way we find stratified pseudomanifolds. We introduce this notion. An open subset W of a stratified space Z is said to be *modeled* on the stratified space L if there exists an isomorphism $\varphi: \mathbb{R}^n \times cL \rightarrow W$. The pair (W, φ) is said to be a *chart* of Z . A family of charts $\{(W, \varphi)\}$, where the family $\{W\}$ is a covering of Z , is called *atlas*.

We shall say that the stratified space Z is a *stratified pseudomanifold* if it there exists a family $\{L_S\}_{S \in \mathcal{S}_Z^{sing}}$ of stratified pseudomanifolds such that for any point $x \in \Sigma_Z$ we can find a chart

(W, φ) modeled on L_S with $\varphi(0, \vartheta) = x$, where S is the stratum of Z containing x . The space L_S is the *link* of the stratum S .

This definition makes sense because is made by induction on the length of Z ($\text{len } L_S < \text{len } Z$). A stratified space with $\text{len } Z = 0$ is always a stratified pseudomanifold. Each of the examples given in 2.1.1 is a stratified pseudomanifold, when Z is a stratified pseudomanifold. This definition is slightly more general than that of stratified pseudomanifold of [7] since we allow the regular stratum to have codimension 1.

2.3 Unfoldings. The computation of the the cohomology of a stratified pseudomanifold Z using differential forms is possible using the controlled forms of Verona [21]; but we need some extra data on Z so that these controlled forms will make sense. The original definition uses a system of neighborhoods of singular strata subjected to some compatibility conditions. A more comprehensive and less technical alternative is presented in [18] where a desingularisation of Z is used. With this blow-up, the controlled forms of M and B are more easily related. In this work we follow this point of view.

Consider Z a stratified pseudomanifold. A continuous map $\mathcal{L}: \tilde{Z} \rightarrow Z$, where \tilde{Z} is a (not necessarily connected) manifold, is an *unfolding* if the two following conditions hold:

1. The restriction $\mathcal{L}_M: \mathcal{L}_M^{-1}(R) \rightarrow R$ is a local diffeomorphism.
2. There exist a family of unfoldings $\{\mathcal{L}_{L_S}: \tilde{L}_S \rightarrow L_S\}_{S \in \mathcal{S}_Z^{\text{sing}}}$ and an atlas \mathcal{A} of Z such that for each chart $(U, \varphi) \in \mathcal{A}$ there exists a commutative diagram

$$\begin{array}{ccc} \mathbb{R}^n \times \tilde{L}_S \times]-1, 1[& \xrightarrow{\tilde{\varphi}} & \mathcal{L}_Z^{-1}(U) \\ Q \downarrow & & \downarrow \mathcal{L}_Z \\ \mathbb{R}^n \times cL_S & \xrightarrow{\varphi} & U \end{array}$$

where

- a) $\tilde{\varphi}$ is a diffeomorphism and
- b) $Q(x_1, \dots, x_n, \tilde{\zeta}, t) = (x_1, \dots, x_n, \llbracket \mathcal{L}_{L_S}(\tilde{\zeta}), |t| \rrbracket)$.

This definition makes sense because is made by induction on the length of Z . When $\text{len } Z = 0$ then \mathcal{L}_Z is just a local diffeomorphism. The restriction $\mathcal{L}_Z: \mathcal{L}_Z^{-1}(S) \rightarrow S$ is a fibration with \tilde{L}_S as a fiber, for any singular stratum S .

For each of the examples of 1.1.1 we have the following unfoldings:

- (a) $\tilde{N} = N$ and $\mathcal{L}_N = \text{identity}$.
- (b) $\tilde{W} = \mathcal{L}_Z^{-1}(W)$ and $\mathcal{L}_W = \text{restriction of } \mathcal{L}_Z$.
- (c) $\widetilde{N \times cZ} = N \times \tilde{Z} \times]-1, 1[$ and $\mathcal{L}_{N \times cZ}(y, \tilde{x}, t) = (y, \llbracket \mathcal{L}_Z(\tilde{x}), |t| \rrbracket)$.

A morphism $f: Y \rightarrow Z$ between two stratified spaces, endowed with unfoldings $\mathcal{L}_Y: \tilde{Y} \rightarrow Y$ and $\mathcal{L}_Z: \tilde{Z} \rightarrow Z$, is a *liftable morphism* if there exists a smooth map $\tilde{f}: \tilde{Y} \rightarrow \tilde{Z}$ with $\mathcal{L}_Z \circ \tilde{f} = f \circ \mathcal{L}_Y$. Each φ is a liftable morphism. The inclusion $W \hookrightarrow Z$ is a liftable morphism.

From now on Z denotes a stratified pseudomanifold endowed with an unfolding $\mathcal{L}_Z: \tilde{Z} \rightarrow Z$.

2.4 Stratifications induced by the action. We present the structure of stratified pseudo-manifold of M and of the orbit space B . For technical reasons we need to consider just in this paragraph a smooth action $\Phi: G \times M \rightarrow M$ of a closed subgroup of the unit circle \mathbb{S}^1 on a manifold M . The properties listed below follow mainly from the Slice Theorem (see [10]).

- *Stratification.* Consider on M the equivalence relation \sim defined by $x \sim y$ iff G_x is equal to G_y , where $G_z = \{g \in G \mid \Phi(g, z) = z\}$ denotes the *isotropy subgroup* of a point $z \in M$. Each of the equivalence class of \sim is an invariant sub-manifold of M . The family \mathcal{S}_M of the connected components of the equivalence classes given by this relation defines a stratification on M . The family $\mathcal{S}_B = \{\pi(S) \mid S \in \mathcal{S}_M\}$ defines a stratification on the orbit space B . When G is connected the map $\pi_{\mathcal{S}}$ is bijective and therefore π is a strict morphism.

We shall write G_S the isotropy subgroup of a point (and therefore any point) of a stratum S . Notice that G_S is a closed subgroup of G . According to this subgroup there are three types of strata: *regular stratum* (G_S is 1), *exceptional stratum* (G_S is finite different from 1) and *fixed stratum* (G_S is \mathbb{S}^1). Notice that the restriction of the canonical projection $\pi: M \rightarrow B$ to S is a principal fibration over $\pi(S)$ with fiber G/G_S . We shall write F the union of fixed strata. We shall identify $F \subset M$ with $\pi(F) \subset B$ by π .

- *Links.* For any singular stratum $S \subset M$ fix a point x on it and put \mathbb{S}^{n_S} the unit sphere of a slice transversal to the stratum S at x . The action Φ induces the orthogonal action $\Phi_S: G_S \times \mathbb{S}^{n_S} \rightarrow \mathbb{S}^{n_S}$; this action has not fixed points (*almost free action*). Notice n_S is necessarily even for a fixed stratum. The link of S is the sphere \mathbb{S}^{n_S} endowed with the stratification induced by Φ_S . The link of $\pi(S)$ is the quotient space \mathbb{S}^{n_S}/G_S . Notice that this link is homologically a sphere or a real projective space when S is an exceptional stratum [2] and a complex projective space when S is a fixed stratum [13].

- *Unfoldings.* It is proven in [10] that M possesses an equivariant unfolding $\mathcal{L}_M: \widetilde{M} \rightarrow M$ (relatively to a free smooth action $\widetilde{\Phi}: G \times \widetilde{M} \rightarrow \widetilde{M}$) in such a way that the induced map $\mathcal{L}_B: \widetilde{M}/G \rightarrow B$ is an unfolding of B . Moreover, if $\widetilde{\pi}: \widetilde{M} \rightarrow \widetilde{M}/G$ is the canonical projection, we have $\mathcal{L}_B \circ \widetilde{\pi} = \pi \circ \mathcal{L}_M$. So, the morphism π is liftable.

3 Controlled forms

Controlled forms were introduced by Verona to compute the cohomology of a stratified pseudo-manifold Z using differential forms [21]. We present them notion in this paragraph, following the approach of [18].

3.1 Definitions. A differential form ω on the regular stratum R of Z is said to be *liftable* if there exists a differential form $\widetilde{\omega}$ on \widetilde{Z} , called the *lifting* of ω , verifying: $\widetilde{\omega} = \mathcal{L}_Z^* \omega$ on $\mathcal{L}_Z^{-1}(R)$. By density the lifting is unique. The differential form ω can be tangent or transversal to the strata; in the first case we get controlled forms and in the second case we get perverse forms.

A liftable form ω is a *controlled form* if it induces a differential form ω_S on each singular stratum S , that is, $\widetilde{\omega}|_{\mathcal{L}_Z^{-1}(S)} = \mathcal{L}_Z^* \omega_S$. So, we can see ω as the family of differential forms $\{\omega_S \in \Omega(S)\}_{S \in \mathcal{S}_Z}$.

We shall write $\Omega_v(Z)$ the *complex of controlled forms* (or the *deRham-Verona complex*). This subcomplex of the deRham complex $\Omega(R)$ is in fact a dgca. To see that, notice that if the differential forms ω and η are controlled, then the differential forms $\omega + \eta$, $\omega \wedge \eta$ and $d\omega$ are also controlled since, for each stratum S , they verify: $\widetilde{\omega + \eta}|_{\mathcal{L}_Z^{-1}(S)} = \widetilde{\omega}|_{\mathcal{L}_Z^{-1}(S)} + \widetilde{\eta}|_{\mathcal{L}_Z^{-1}(S)} = \omega_S + \eta_S$, $\widetilde{\omega \wedge \eta}|_{\mathcal{L}_Z^{-1}(S)} = \widetilde{\omega}|_{\mathcal{L}_Z^{-1}(S)} \wedge \widetilde{\eta}|_{\mathcal{L}_Z^{-1}(S)} = \omega_S \wedge \eta_S$, and $\widetilde{d\omega}|_{\mathcal{L}_Z^{-1}(S)} = d\widetilde{\omega}|_{\mathcal{L}_Z^{-1}(S)} = d\omega_S$.

For a stratum S , we have the restriction operator $R_S: \Omega_v(Z) \rightarrow \Omega(S)$, defined by $R_S(\omega) = \omega_S$, which is a dgca operator and therefore endows $\Omega(S)$ with a structure of $\Omega_v(Z)$ -dg module.

Proposition 3.1.1 *When S is closed the restriction operator $R_S: \Omega_v(Z) \rightarrow \Omega(S)$ is onto.*

Proof. Fix (U, ϕ) a chart of \mathcal{A} . Consider $\alpha: \mathbb{R}^n \rightarrow \mathbb{R}$ and $\beta:]-1, 1[\rightarrow \mathbb{R}$ two smooth maps taking the value 1 on a neighborhood of 0. The map $f: Z - \Sigma \rightarrow \mathbb{R}$ defined from $f\phi(x_1, \dots, x_n, [\zeta, t]) = \alpha(x_1, \dots, x_n) \cdot \beta(t)$ is a controlled form. In fact, its lifting is the smooth map $\tilde{f}: \tilde{Z} \rightarrow \mathbb{R}$ defined from $\tilde{f}\tilde{\phi}(x_1, \dots, x_n, [\tilde{\zeta}, t]) = \alpha(x_1, \dots, x_n) \cdot \beta(|t|)$ which is constant on the fibers of \mathcal{L}_Z . A standard argument shows that there exists a partition of unity subordinated to \mathcal{A} .

We reduce the problem to show that $R_S: \Omega_v(U) \rightarrow \Omega(U \cap S)$ is onto. Since S is closed then $U \cap S$ is the lowest stratum of U and therefore the question becomes: the restriction operator $R_{\mathbb{R}^n}: \Omega_v(\mathbb{R}^n \times cL_S) \rightarrow \Omega(\mathbb{R}^n)$ is onto? And the answer is clearly yes. ♣

3.2 Relative controlled forms Consider Y a union of strata of Z . A *relative controlled form* on (Z, Y) is a controlled form on Z vanishing on Y , that is, $\omega_S \equiv 0$ for each stratum $S \subset Y$. We shall write $\Omega_v(Z, Y)$ the complex of relative controlled forms, which is a dgca. If we consider

the restriction operator $R_Y = \prod_{S \subset Y} R_S: \Omega_v(Z) \rightarrow \Omega\left(Y = \bigsqcup_{S \subset Y} S\right) = \prod_{S \subset Y} \Omega(S)$ then we can write $\Omega_v(Z, Y) = \text{Ker } R_Y$.

The wedge product $\wedge: \Omega_v(Z) \times \Omega_v(Z, Y) \longrightarrow \Omega_v(Z, Y)$ endows $\Omega_v(Z, Y)$ with a structure of $\Omega_v(Z)$ -dg module. The natural inclusion $\Omega_v(Z, Y) \hookrightarrow \Omega_v(Z)$ is a morphism in the category of $\Omega_v(Z)$ -dg modules.

3.3 Controlled model. The deRham-Verona complex $\Omega_v(Z)$ depends on the unfolding chosen, but for a stratified space its cohomology does not: $H(\Omega_v(Z)) \cong H(Z)$, the singular cohomology with real coefficients (cf. [21], [18] where controlled becomes zero perversity). But we have a stronger result at the level of dgca minimal models. Recall that the dgca minimal model $\mathcal{A}(Z)$ of Z is just the dgca minimal model of $A_{\mathbb{R}}(Z)$, the dgca of polynomials differential forms on the simplicial set $\text{Sing}(Z)$ of singular simplices of Z . The dgca $\Omega_v(Z)$ is easier to handle than $A_{\mathbb{R}}(Z)$, but we need to know that their dgca minimal models are the same. This follows immediately from the following Theorem (cf. [8]).

Theorem 3.3.1 *Let Z be a stratified space. Then there exists two quasis of dgca algebras*

$$\Omega_v(Z) \xrightarrow{\mathbb{N}_1} \cdot \xleftarrow{\mathbb{N}_2} A_{\mathbb{R}}(Z).$$

Proof. See Appendix. ♣

3.4 Relative controlled model. As in the absolute case (cf. §3.3.1), the model of a morphism $f: Z' \rightarrow Z$ between two stratified spaces can be computed, under certain conditions, using controlled forms instead polynomial forms. These conditions are

[P1] f preserves controlled forms: $f^*\omega \in \Omega_v(Z')$ for any $\omega \in \Omega_v(Z)$

[P2] f preserves liftable simplices: $f \circ \varphi'$ -decomposition = φ' -decomposition and

$f \circ \varphi' \in \underline{LS}(Z)$ for any $\varphi' \in \underline{LS}(Z')$.

We shall say that f satisfying the two conditions is *good*.

For good morphisms at least, we have a relative version of Theorem 3.3.1.

Theorem 3.4.1 *Let $f: Z' \rightarrow Z$ be a good morphism of stratified spaces. Then there exists a commutative diagram of dgca morphisms*

$$\begin{array}{ccccc} \Omega_v(Z') & \xrightarrow{\aleph'_1} & \cdot & \xleftarrow{\aleph'_2} & A_{\mathbb{R}}(Z') \\ \uparrow f^* & & \uparrow & & \uparrow f^* \\ \Omega_v(Z) & \xrightarrow{\aleph_1} & \cdot & \xleftarrow{\aleph_2} & A_{\mathbb{R}}(Z) \end{array}$$

in which the horizontal arrows are quis.

Proof. See Appendix. ♣

The two examples of good morphisms used in this work are described in, the following proposition.

Lemma 3.4.2 *Let $\varphi: \Delta \rightarrow M$ be a simplex satisfying [LS1]. Then, the family of strata of M meeting $\text{Im } \varphi$ is totally ordered.*

Proof. We prove that if F_1, F_2 are two faces of Δ and S_1, S_2 two strata of M with $\varphi(\text{int}(F_i)) \cap S_i \neq \emptyset$, $i = 1, 2$, then $S_1 \leq$ or $S_2 \leq S_1$.

Since $\varphi^{-1}(M_{\dim S_i})$ (resp. $\varphi^{-1}(M_{\dim S_i - 1})$) is a face of Δ meeting $\text{int}(F_i)$ (resp. not containing $\text{int}(F_i)$) then $\text{int}(F_i) \subset \varphi^{-1}(M_{\dim S_i})$ (resp. $\text{int}(F_i) \cap \varphi^{-1}(M_{\dim S_i}) = \emptyset$). So, $\text{int}(F_i) \subset \varphi^{-1}(M_{\dim S_i} - (M_{\dim S_i - 1}))$ and by connectivity we get $\text{int}(F_i) \subset \varphi^{-1}(S_i)$. Notice that this implies $\varphi(F_i) \subset \overline{S_i}$.

Consider now F_3 the smallest face of Δ containing F_1 and F_2 . Put S_3 a stratum of M with $\varphi(\text{int}(F_3)) \cap S_3 \neq \emptyset$ (it always exists!). From the previous paragraph we get $\varphi(F_3) \subset S_3$ and $\varphi(F_3) \subset \overline{S_3}$ and therefore $S_1 \cap \overline{S_3} \neq \emptyset$, $S_2 \cap \overline{S_3} \neq \emptyset$ and so $S_1 \leq S_3$ and $S_2 \leq S_3$.

Let us suppose $\dim S_1 \leq \dim S_2$. Since the face $\varphi^{-1}(M_{\dim S_2})$ contains $\text{int}(F_1)$ and $\text{int}(F_2)$ then it also contains F_3 by minimality. Thus $S_3 \subset M_{\dim S_2}$ which gives $\dim S_3 \leq \dim S_2$ and therefore $S_3 = S_2$. Finally $S_1 \leq S_2$. ♣

Proposition 3.4.3 *The projection $\pi: M \rightarrow B$ and the inclusion $\iota: F \hookrightarrow B$ are good morphisms.*

Proof. We need to verify conditions [P1] and [P2].

• **Projection.**

[P1] Since π is a liftable morphism.

[P2] Let $\varphi: \Delta \rightarrow M$ be a liftable simplex. Following the lemma the family of strata meeting $\text{Im } \varphi$ can be written $S_0 < S_1 < \dots < S_{p-1} < S_p$. The φ -decomposition of Δ is $\Delta = \Delta_0 * \dots * \Delta_p$ with $\Delta_0 * \dots * \Delta_i = \varphi^{-1}(S_0 \cup \dots \cup S_i)$. Since π is a strict morphism then the family of strata of B meeting $\text{Im}(\pi \circ \varphi)$ is $\pi(S_0) < \pi(S_1) < \dots < \pi(S_{p-1}) < \pi(S_p)$. So, $\Delta_0 * \dots * \Delta_i = (\pi \circ \varphi)^{-1}(\pi(S_0) \cup \dots \cup \pi(S_i))$ which implies that the $(\pi \circ \varphi)$ -decomposition of Δ is the φ -decomposition. The lifting of $\pi \circ \varphi$ is just $\tilde{\pi} \circ \tilde{\varphi}$.

• **Inclusion.**

[P1] Since $\text{len}(F) = 0$ then we have $\Omega_v(F) = \Omega(F)$. Notice that $\iota^*: \Omega_v(B) \rightarrow \Omega(F)$ is just R_F .

[P2] Consider $\varphi: \Delta \rightarrow F$ a liftable simplex, that is, a smooth map $\varphi: \Delta \rightarrow S$ where S is a fixed stratum. So, the φ -decomposition and the $(\iota \circ \varphi)$ -decomposition are just $\Delta = \Delta$. On the other hand, since $\mathcal{L}_B: \mathcal{L}_B^{-1}(S) \rightarrow S$ is a fibration and Δ is contractible then we can construct a smooth map $\psi: \Delta \rightarrow \tilde{Z}$ with $\mu_Z \circ \psi = \iota \circ \varphi$. So, this map is the lifting of $\iota \circ \varphi$. ♣

4 Models for M and F

This section is devoted to construct the minimal models of M and F in the category of differential graduate models over the dgca minimal model $\mathcal{A}(B)$ of B . As it happens in other contexts (Euler-Poincaré characteristic, Localization Theorem, rational homotopy theory,...), these two models are intimately related, in fact, they are free $\mathcal{A}(B)$ -graded modules over the same (up to a shift) graded vectorial space.

4.1 Model of a stratified space. Let Z be a stratified space. As we showed in Theorem 3.3.1, its dgca minimal model can be computed from $\Omega_v(Z)$. So let

$$\rho_Z: \mathcal{A}(Z) \xrightarrow{\simeq} \Omega_v(Z)$$

be a quis of dgc algebras, with $\mathcal{A}(Z)$ a minimal one. Next, let $f: Z' \rightarrow Z$ be a good morphism. Then we can endow $\Omega_v(Z')$ with a structure of $\mathcal{A}(Z)$ -dg module by means of the composition

$$\mathcal{A}(Z) \xrightarrow{\rho_Z} \Omega_v(Z) \xrightarrow{f^*} \Omega_v(Z').$$

Since we will always consider this structure of module in $\Omega_v(Z')$ we shall not write $\rho_Z^* f^*(\Omega_v(Z'))$ but simply $\Omega_v(Z')$. In the same way, we also consider $A_{\mathbb{R}}(Z')$ as an $\mathcal{A}(Z)$ -dg module.

Proposition 4.1.1 *The $\mathcal{A}(Z)$ -dgm minimal models of $\Omega_v(Z')$ and $A_{\mathbb{R}}(Z')$ are isomorphic.*

Proof. It follows from Theorem 3.4.1. The commutativity of the diagram given by this result means that \aleph'_1 and \aleph'_2 are quis of $\mathcal{A}(Z)$ -dg modules. ♣

Let us denote by $\mathcal{M}(Z')$ this $\mathcal{A}(Z)$ -dgm minimal model. It obviously depends on f but not on the several choices we have made in this construction: the dgc algebra $\mathcal{A}(Z)$, the quis ρ_Z and the $\mathcal{A}(Z)$ -dgm minimal model of $\Omega_v(Z')$ (or $A_{\mathbb{R}}(Z')$). Despite of all these choices, the $\mathcal{A}(Z)$ -dgm minimal model $\mathcal{M}(Z')$ is unique up to isomorphism by section § 1.3.

4.2 Minimal model of F . The fixed point set F plugs into the category of $\mathcal{A}(B)$ -dg modules through the natural inclusion $\iota: F \hookrightarrow B$ which is a good morphism. We have already seen in § 3.2 that $\Omega_v(B, F)$ is an $\Omega_v(B)$ -dg module and therefore an $\mathcal{A}(B)$ -dg module. We shall write $\mathcal{M}(B, F)$ the *relative minimal model* of (B, F) , that is, the $\mathcal{A}(B)$ -dgm minimal model of $\Omega_v(B, F)$. Notice that the inclusion $i: \Omega_v(B, F) \hookrightarrow \Omega_v(B)$ is an $\mathcal{A}(B)$ -dgm morphism. We shall write $i': \mathcal{M}(B, F) \rightarrow \mathcal{A}(B)$ any of its models (see Corollary 1.3.6). The degree of i and i' is 0.

Proposition 4.2.1

$$(4) \quad \mathcal{M}(F) = \mathcal{A}(B) \oplus_{i'} \mathcal{M}(B, F).$$

Proof. Consider the exact sequence $0 \rightarrow \Omega_v(B, F) \xrightarrow{i} \Omega_v(B) \xrightarrow{R_F} \Omega(F) \rightarrow 0$ (cf. Proposition 3.1.1). Then, by Remark 1.1.1 we have a $\mathcal{A}(B)$ -dg module quis between $\Omega_v(B) \oplus_i \Omega_v(B, F)$ and $\Omega(F)$. So, by Proposition 1.3.7 we have a $\mathcal{A}(B)$ -dgm quis between $\mathcal{A}(B) \oplus_{i'} \mathcal{M}(B, F)$ and $\Omega(F)$. Since $\mathcal{A}(B) \oplus_{i'} \mathcal{M}(B, F)$ is a minimal $\mathcal{A}(B)$ -dgm (cf. Lemma 1.3.8) then we get by uniqueness $\mathcal{M}(F) = \mathcal{A}(B) \oplus_{i'} \mathcal{M}(B, F)$ (cf. Corollary 1.3.5). \clubsuit

Corollary 4.2.2 *Suppose B of finite type and simply connected, then $\mathcal{M}(B, F)$ is the free $\mathcal{A}(B)$ -graded module generated by $\tilde{H}^{*-1}(Y_i)$.*

Proof. From § 1.3.7 we know that $\mathcal{M}(F)$ is a free $\mathcal{A}(B)$ -graded module over $H^*(Y_i)$ and therefore $\mathcal{M}(F) = \mathcal{A}(B) \oplus_h [\mathcal{A}(B) \otimes \tilde{H}(Y_i)]$ for some h of degree 1. From the above Proposition we get the result wanted. \clubsuit

4.2.3 Remarks

- M of finite type $\Rightarrow B$ of finite type. From [3] we know that if M is of finite type then F and (B, F) are of finite type. Using the long exact sequence associated to (B, F) one gets that B is also of finite type.

- M simply connected $\Rightarrow B$ simply connected. Considering twisted neighborhoods of orbits (cf. [3]) one easily checks that a loop on B lifts in a path on M . Since the orbits of M are connected we conclude.

4.3 Minimal model of M . The fundamental vector field X of the action is defined by $X(x) = T_1\Phi_x(1)$, where $x \in M$ and $\Phi_x: \mathbb{S}^1 \rightarrow M$ is given by $\Phi_x(g) = \Phi(g, x)$. Since this vector field does not vanish on R we can consider the dual form $\chi \in \Omega^1(R)$, relatively a riemannian metric μ on R . When this metric is *good* (cf. [10]) the derivative $d\chi$ is a basic form relatively to the projection $\pi: R \rightarrow \pi(R)$. So, there exists a differential form $e \in \Omega^2(\pi(R))$ verifying $d\chi = \pi^*e$. Both differential forms, χ and e , are liftable. We shall say that e is an *Euler form*. They are not controlled forms because their restriction to the links of fixed strata do not vanish necessarily. But the maps $\chi: \Omega_v(M, F) \rightarrow \Omega_v(M)$ and $e: \Omega_v(B, F) \rightarrow \Omega_v(B)$, given by $\gamma \mapsto \chi \wedge \gamma$ and $\omega \mapsto e \wedge \omega$, are well defined $\mathcal{A}(B)$ -dgm morphisms. We shall write $e': \mathcal{M}(B, F) \rightarrow \mathcal{A}(B)$ a model of e . Notice that the degree of e and e' is 2.

The main result of this work is

Theorem 4.3.1

$$(5) \quad \mathcal{M}(M) = \mathcal{A}(B) \oplus_{e'} \mathcal{M}(B, F).$$

Proof. Put $I\Omega_v(M) = \{\omega \in \Omega_v(M) / L_X\omega j = 0\}$ the complex of invariant controlled differential forms. We have seen in [10] that the inclusion $I\Omega_v(M) \hookrightarrow \Omega_v(M)$ is a dgca quis. We endow $I\Omega_v(M)$ with the natural structure of $\mathcal{A}(Z)$ -dg module by means of the composition

$$\mathcal{A}(B) \xrightarrow{\rho_B} \Omega_v(B) \xrightarrow{\pi^*} \Omega_v(M),$$

which is well defined since $\pi^*(\Omega_v(B)) \subset I\Omega_v(M)$. The inclusion $I\Omega_v(M) \hookrightarrow \Omega_v(M)$ is now a $\mathcal{A}(B)$ -dgm quis. Each invariant differential form ω is written uniquely as $\omega = \pi^*\alpha + \chi \wedge \pi^*\beta$;

when ω is controlled then $\alpha \in \Omega_v(B)$ and $\beta \in \Omega_v(B, F)$ because X is tangent to the links of fixed strata. So, the operator

$$(6) \quad \Delta: \Omega_v(B) \oplus_e \Omega_v(B, F) \longrightarrow I\Omega_v(M),$$

given by $\Delta(\alpha, \beta) = \pi^*\alpha + \chi \wedge \pi^*\beta$ is an $\mathcal{A}(B)$ -dgm isomorphism and therefore $\mathcal{A}(B) \oplus_{e'} \mathcal{M}(B, F)$ is a $\mathcal{A}(B)$ -dgm model of M (cf. Proposition 1.3.7). This model is minimal because of Lemma 1.3.8. ♣

4.3.2 Remarks.

a) Formulae (4) and (5) show that $\mathcal{M}(F)$ and $\mathcal{M}(M)$ are free $\mathcal{A}(B)$ -graded modules over the same (up to a shift by 2) basis.

b) This theorem contains the classic result saying that, when the action is *almost free* (that is, $F = \emptyset$), the dgca minimal model of M is $\mathcal{A}(B) \otimes \Lambda(x)$ with $\deg x = 1$ (cf. [6]). Let us see that.

First of all, a dgca quis between $\mathcal{A}(B) \otimes \Lambda(x)$ and $\Omega_v(M)$ is given by $\Theta(a \otimes 1 + b \otimes x) = \pi^*\rho_B(a) + \pi^*\rho_B(b) \wedge \chi$. Next, since we can choose a model of e to be the product by some element $e' \in \mathcal{A}(B)$ with $[\rho_B(e')] = [e]$, we have $\mathcal{M}(M) = \mathcal{A}(B) \oplus_{e'} \mathcal{A}(B) \cong \mathcal{A}(B) \otimes \Lambda(x)$ (as $\mathcal{A}(B)$ -dg modules). After composing with this isomorphism, the $\mathcal{A}(B)$ -dgm quis $\Delta \circ (\rho_B \oplus \rho_B)$ becomes Θ . So, $\mathcal{M}(M) = \mathcal{A}(B) \otimes \Lambda(x)$ as dgca algebras.

We establish now some consequences of these results.

4.4 Poincaré polynomial. Given a topological space X we shall write P_X its Poincaré polynomial, that is, $P_X(t) = \sum_{n \geq 0} \dim H^n(X) \cdot t^n$.

Corollary 4.4.1 *Suppose B of finite type and simply connected, then $1 - P_{Y_\pi} = t^2(1 - P_{Y_\iota})$.*

Proof. We have seen in Proposition 4.2.2 that $\mathcal{M}(B, F)$ is a free $\mathcal{A}(B)$ -graded module over $\tilde{H}^{*-1}(Y_\iota)$. Applying the same method to π we get that $\mathcal{M}(B, F)$ is a free $\mathcal{A}(B)$ -graded module over $\tilde{H}^{*+1}(Y_\pi)$. So, $\dim H^0(Y_\pi) = 1$, $\dim H^1(Y_\pi) = 0$, $\dim H^2(Y_\pi) = \dim H^0(Y_\iota) - 1$ and $\dim H^n(Y_\pi) = \dim H^{n-2}(Y_\iota)$ for $n \geq 2$. This gives the result. ♣

4.5 Vanishing the Euler class. Actions with vanishing Euler class $[e] \in H^2(B - F)$ have a particular status (cf. [10] for a geometrical interpretation). We show in the sequel how $\mathcal{M}(M)$ contains information about the dgca minimal model of M in this case.

When the Euler class vanishes we can choose a convenient riemannian metric on M so that e itself vanishes (cf. [10]). Thus $e' = 0$. The minimal model $\mathcal{M}(M)$ is of the form $\mathcal{A}(B) \otimes E$ with $E^0 = \mathbb{R}$ and $dE \subset \mathcal{A}(B) \otimes E^+$. It supports a dgca structure by putting on E the trivial product: $1 \cdot v = v$ if $v \in E$ and $E^+ \cdot E^+ = 0$. This dgca structure shall be said *naïve*. It contains the following information about M .

Corollary 4.5.1 *If the Euler class vanishes then the naïve dgca structure of $\mathcal{M}(M)$ has the same real homotopy type of M . Moreover, $\pi_\psi^*(B)$ injects into $\pi_\psi^*(M)$.*

Proof. The operator $\Delta: \Omega_v(B) \oplus_0 \Omega_v(B, F) \longrightarrow \Omega_v(M)$ is an $\mathcal{A}(B)$ -dgm quis which becomes a dgca quis when considering on the source the following product:

$$(7) \quad (\alpha, \beta) \cdot (\alpha', \beta') = (\alpha \cdot \alpha', (-1)^{\deg \alpha} \alpha \cdot \beta' + (-1)^{\deg \alpha' \cdot \deg \beta} \alpha' \cdot \beta).$$

This dgca contains the real homotopy type of M .

Put $\rho_{(B,F)}: \mathcal{M}(B, F) \rightarrow \Omega_v(B, F)$ the relative minimal model of (B, F) . The operator

$$\rho_B \oplus \rho_{(B,F)}: \mathcal{M}(M) = \mathcal{A}(B) \oplus_0 \mathcal{M}(B, F) \longrightarrow \Omega_v(B) \oplus_0 \Omega_v(B, F)$$

is an $\mathcal{A}(B)$ -dgm quis which becomes a dgca quis when considering in both terms the product (7). This dgca contains the real homotopy type of M . Notice that the dgca structure on $\mathcal{M}(M)$ given by (7) is just the naïve structure. This gives the first part of the Corollary.

We shall write $\mathcal{A}(B) = \Lambda Y$, with differential ∂ , and $\mathcal{M}(M) = \Lambda Y \otimes E$, with differential d . We have $d|_{\Lambda Y} = \partial$ and $dE \subset \Lambda Y \otimes E^+$. This last property allows us to construct a KS-extension

$$\begin{array}{ccc} (\Lambda Y, \partial) & \xrightarrow{\quad} & (\Lambda Y \otimes E, d) \\ & \searrow & \uparrow \varphi \\ & & (\Lambda Y \otimes \Lambda X, \delta) \end{array}$$

verifying $\varphi|_{\Lambda Y} = \text{id}_{\Lambda Y}$ and $\varphi(X) \subset \Lambda Y \otimes E^+$. Recall that $\pi_\psi^*(B) = H(Y, \partial_0)$ and $\pi_\psi^*(M) = H(Y \oplus X, \delta_0)$, where ∂_0 and δ_0 are the linear part of ∂ and δ respectively.

Since $(\Lambda Y, \partial)$ is minimal we just have $\partial_0 = 0$ and therefore $\pi_\psi^*(B) = Y$. On the other hand, the composition $\delta_0 = X \xrightarrow{\delta} \Lambda Y \otimes \Lambda X \xrightarrow{\text{pro}} \Lambda Y \xrightarrow{\text{pro}} Y$ vanishes: if $x \in X$ then $\varphi(\delta x) = d(\varphi x) \in d(\Lambda Y \otimes E^+) \subset \Lambda Y \otimes E^+$. So, $\pi_\psi^*(M) = Y \oplus H(X, \delta_0)$ and the proof is ended since φ^* becomes the inclusion $Y \hookrightarrow Y \oplus H(X, \delta_0)$. ♣

4.5.2 Remarks.

a) When B is also contractible then $\mathcal{M}(M) = E$ is just a dgca with trivial product and, by minimality, with zero derivative. In other words, M is a wedge of spheres.

b) The Gysin sequence associated to (5) implies that the cohomology of B injects into the cohomology of M , in fact, we have the short exact sequence

$$0 \rightarrow H^*(B) \rightarrow H^*(M) \rightarrow H^{*-1}(B, F) \rightarrow 0.$$

The last statement of the Corollary implies that, when M is of finite type and simply connected, we have the following short exact sequence

$$0 \rightarrow \pi^*(B) \otimes \mathbb{R} \rightarrow \pi^*(M) \otimes \mathbb{R} \rightarrow \pi^*(Y_\pi) \otimes \mathbb{R} \rightarrow 0.$$

c) The naïve structure of $\mathcal{M}(M)$ appears when $e' = 0$, but the previous result needs the vanishing of the Euler class itself as is shown in the following example. Consider the action $\Phi: \mathbb{S}^1 \times \mathbb{CP}^n \rightarrow \mathbb{CP}^n$ given by $z \cdot [z_0, z_1, \dots, z_n] = [z_0, z \cdot z_1, \dots, z \cdot z_n]$ (in homogeneous coordinates). Here the fixed point set is $F = \mathbb{CP}^0 \cup \mathbb{CP}^{n-1}$ and the orbit space B is the closed cone over \mathbb{CP}^{n-1} . So, B is acyclic and $e' = 0$. The Euler class does not vanish since it generates $H^2(B - F) = H^2(\mathbb{CP}^{n-1} \times]0, 1]) = \mathbb{R}$. The minimal model we have computed in Theorem 4.3.1 is $\mathcal{M}(\mathbb{CP}^n) = \mathbb{R} \oplus_0 \tilde{H}^{*-1}(\mathbb{CP}^0 \cup \mathbb{CP}^{n-1})$. Considering on it the naïve structure we get $\mathcal{M}(\mathbb{CP}^n) = H^*(\mathbb{S}^2 \vee \mathbb{S}^4 \vee \dots \vee \mathbb{S}^{2n})$ as dgc algebras. But clearly this dgca algebra does not contain the real homotopy type of \mathbb{CP}^n .

4.6 Cohomological dimension. Write $\dimc(X)$ the cohomological dimension of the topological space X , that is, $\dimc(X) = \sup\{n \in \mathbb{N} / H^n(X) \neq 0\}$.

Corollary 4.6.1 *Under the hypothesis of Corollary 4.4.1, if $\dimc(B)$ and $\dimc(Y_\pi)$ are finite then $\dimc(M) = \dimc(F)$ or $\dimc(F) + 2$.*

Proof. From Corollary 4.4.1 we get $\dimc(Y_\pi) = \dimc(Y_i) = 0$ or $\dimc(Y_\pi) = \dimc(Y_i) + 2$. Considering now the homotopy fibrations associated to π and ι we get $\dimc(M) = \dimc(B) + \dimc(Y_\pi)$ and $\dimc(F) = \dimc(B) + \dimc(Y_i)$ and then we get the result. \clubsuit

We find examples of this situation when $M = \mathbb{R}^{n+2} = \mathbb{S}^1 * \mathbb{R}^n$ and $M = \mathbb{S}^{n+2} = \mathbb{S}^1 * \mathbb{S}^n$ where \mathbb{S}^1 acts by multiplication on the first factor and trivially on the second factor. When M is compact an oriented the condition $\dimc(M) = \dimc(F) = 0$ does not occur and the condition $\dimc(M) = \dimc(F) + 2$ is equivalent to say that F possesses a connected component of codimension 2. This is also equivalent to the fact that B has a boundary. So, under the conditions of Corollary 4.4.1, if $\dimc(B) < \infty$ and B has not boundary then $\dimc(Y_\pi) = \dimc(Y_i) = \infty$.

4.7 Equivariant cohomology.

The *equivariant cohomology* of M is the cohomology of the quotient space $M_{\mathbb{S}^1} = M \times_{\mathbb{S}^1} \mathbb{S}^\infty$, written $H_{\mathbb{S}^1}(M)$. The natural projection $p: M_{\mathbb{S}^1} \rightarrow B$ induces a natural structure of $\mathcal{A}(B)$ -dgm module on $M_{\mathbb{S}^1}$. We compute here the *equivariant minimal model* of M , that is, $\mathcal{M}_{\mathbb{S}^1}(M) = \mathcal{M}(M_{\mathbb{S}^1})$.

We shall write $\Lambda(e)$ the exterior algebra generated by an element e of degree 2. The trivial $\mathcal{A}(B)$ -dgm structure will be considered on it. The main result in this framework is

Theorem 4.7.1 *If the fixed point set F is not empty, then*

$$(8) \quad \mathcal{M}_{\mathbb{S}^1}(M) = [\mathcal{A}(B) \otimes \Lambda(e)] \oplus_{q'} [\mathcal{M}(B, F) \otimes \Lambda(e)],$$

where $q'(b \otimes e^n) = e'(b) \otimes e^n + i'(b) \otimes e^{n+1}$.

Proof. The equivariant cohomology can be computed using the complex $\Omega_{\mathbb{S}^1}(M) = I\Omega(M) \otimes \Lambda(e)$, endowed with the derivative $d(\omega \otimes e^n) = (d\omega) \otimes e^n + (i_X \omega) \otimes e^{n+1}$. Here i_X denotes the contraction by X . Proceeding as in [18, p.213] one shows that the two dgc algebras $I\Omega(M)$ and $I\Omega_v(M)$ are quasi-isomorphic. Therefore the equivariant cohomology of M is computed by using $I\Omega_v(M) \otimes \Lambda(e)$ and p induces the operator $P: \Omega_v(B) \rightarrow I\Omega_v(M) \otimes \Lambda(e)$ defined by $P(\alpha) = \pi^* \alpha \otimes 1$. Under these transformations the $\mathcal{A}(B)$ -dgm structure on $I\Omega_v(M) \otimes \Lambda(e)$ is given by:

$$a \cdot (\omega \otimes e^n) = P(\rho_B(a)) \cdot (\omega \otimes e^n), \quad \text{with } a \in \mathcal{A}(B), \omega \in I\Omega_v(M).$$

We compute now the minimal model of $I\Omega_v(M) \otimes \Lambda(e)$ relatively to the this structure.

The $\mathcal{A}(B)$ -dgm isomorphism $\Delta: \Omega_v(B) \oplus_e \Omega_v(B, F) \rightarrow I\Omega_v(M)$ induces the $\mathcal{A}(B)$ -dgm isomorphism

$$\nabla: [\Omega_v(B) \otimes \Lambda(e)] \oplus_q [\Omega_v(B, F) \otimes \Lambda(e)] \rightarrow I\Omega_v(M) \otimes \Lambda(e),$$

where $q(\beta \otimes e^n) = (\beta \wedge e) \otimes e^n + i(\beta) \otimes e^{n+1}$. Proceeding as in Theorem 4.3.1 we get that a model of $I\Omega_v(M) \otimes \Lambda(e)$ is just $[\mathcal{A}(B) \otimes \Lambda(e)] \oplus_{q'} [\mathcal{M}(B, F) \otimes \Lambda(e)]$, where $q'(b \otimes e^n) = e'(b) \otimes e^n + i'(b) \otimes e^{n+1}$. This model is minimal because of Lemma 1.3.8. \clubsuit

Notice that for the almost free case ($F = \emptyset$) we have obtained the following non minimal $\mathcal{A}(B)$ -dgm model: $[\mathcal{A}(B) \otimes \Lambda(e)] \oplus_{i' \otimes e} [\mathcal{A}(B) \otimes \Lambda(e)]$; the minimal one is just $\mathcal{A}(B)$.

4.7.2 Remarks

a) *Poincaré polynomial.* When $F \neq \emptyset$, B is of finite type and simply connected the relation between the homotopy fibers of p and π is given by $P_{Y_\pi} = (1 - t^2)P_{Y_p}$ (same proof as that of Corollary 4.4.1).

b) *Extension of scalars.* The complexes $\Omega_{\mathbb{S}^1}(M)$ and $\mathcal{M}_{\mathbb{S}^1}(M)$ support naturally a structure of $\Lambda(e)$ -dgm module. The $\mathcal{A}(B)$ -dgm quis we have constructed is in fact a $\Lambda(e)$ -dgm quis. For this structure the extension of scalars of $\mathcal{M}_{\mathbb{S}^1}(M)$ is just $\mathcal{M}(M)$, that is, $\mathbb{R} \otimes_{\Lambda(e)} \mathcal{M}_{\mathbb{S}^1}(M) = \mathcal{M}(M)$. In other words, the model of the fiber of $M \rightarrow M_{\mathbb{S}^1} \rightarrow B$ is "the fiber of the model" (cf. [8]).

c) *Vanishing Euler class.* Since $e' = 0$ one gets that $\mathcal{M}_{\mathbb{S}^1}(M)$ is isomorphic to $\mathcal{A}(B) \oplus [\mathcal{M}(F) \otimes \Lambda^+(e)]$, relatively to both module structures. We conclude that the cohomology of B injects into the equivariant cohomology of M , that is, we have in fact the following exact sequence $0 \rightarrow H(B) \rightarrow H_{\mathbb{S}^1}(M) \rightarrow H(F) \otimes \Lambda^+(e) \rightarrow 0$.

d) *Equivariant cohomology.* Formula (8) says that we can compute the equivariant minimal model $\mathcal{M}_{\mathbb{S}^1}(M)$ in terms of basic data: $i', e': \mathcal{M}(B, F) \rightrightarrows \mathcal{A}(B)$. The equivariant cohomology $H_{\mathbb{S}^1}^*(M)$ can be also computed in terms of basic data. In fact, the short exact sequence $0 \rightarrow \mathcal{A}(B) \otimes \Lambda(e) \rightarrow \mathcal{M}_{\mathbb{S}^1}(M) \rightarrow \mathcal{M}(B, F) \otimes \Lambda(e) \rightarrow 0$ associated to (8) (cf. Lemma 1.1.2) gives the long exact sequence

$$\cdots \rightarrow [H(B) \otimes \Lambda(e)]^i \rightarrow H_{\mathbb{S}^1}^i(M) \rightarrow [H(B, F) \otimes \Lambda(e)]^{i-1} \xrightarrow{q^*} [H(B) \otimes \Lambda(e)]^{i+1} \rightarrow \cdots,$$

which determines $H_{\mathbb{S}^1}(M)$ in terms of $i^*, e^*: H(B, F) \rightrightarrows H(B)$.

e) *Equivariantly formal spaces.* Put $\mathbf{r}: M \rightarrow M_{\mathbb{S}^1}$ the inclusion given by $\mathbf{r}(x) = \text{class of } (x, 1)$. The manifold M is *equivariantly formal* if the restriction map $\mathbf{r}^*: H_{\mathbb{S}^1}(M) \rightarrow H(M)$ is surjective (cf. [3], [4]). We can translate this condition in terms of basic data by considering the following commutative diagram

$$\begin{array}{ccccccc} \cdots \rightarrow & H_{\mathbb{S}^1}^i(M) & \longrightarrow & [H(B, F) \otimes \Lambda(e)]^{i-1} & \xrightarrow{q^*} & [H(B) \otimes \Lambda(e)]^{i+1} & \rightarrow \cdots \\ & \downarrow \mathbf{r}^* & & \downarrow \mathfrak{R} & & \downarrow \mathfrak{R} & \\ \cdots \rightarrow & H^i(M) & \longrightarrow & H^{i-1}(B, F) & \xrightarrow{e^*} & H^i(B) & \longrightarrow \cdots \end{array}$$

where $\mathfrak{R}(\sum [\alpha_n] \otimes e^n) = [\alpha_n]$. Since the restriction $\mathfrak{R}: \text{Coker } q^* \rightarrow \text{Coker } e^*$ is surjective then the manifold M is equivariantly formal if and only if the restriction $\mathfrak{R}: \text{Ker } q^* \rightarrow \text{Ker } e^*$ is surjective. In other words, any string $[\alpha_0] \xrightarrow{e^*} 0$ fits into a string $0 \xleftarrow{i^*} [\alpha_n] \xrightarrow{e^*} [\beta_n] \xleftarrow{i^*} \cdots \xrightarrow{e^*} [\beta_0] \xleftarrow{i^*} [\alpha_0] \xrightarrow{e^*} 0$.

f) *Localization Theorem.* This Theorem asserts that the restriction map $R_F: M \rightarrow F$ induces an isomorphism between their localizations $S^{-1}H_{\mathbb{S}^1}(M)$ and $S^{-1}H_{\mathbb{S}^1}(F)$. We can translate this Theorem in terms of basic data saying that the map

$$\nabla: H(B, F; S^{-1}\Lambda(e)) \longrightarrow H(B, F; S^{-1}\Lambda(e)),$$

defined from $\nabla([\omega]) = e \cdot [\omega] + [\omega \wedge e]$, is an isomorphism. This comes from the fact that the Localization Theorem is equivalent to the vanishing of $S^{-1}H_{\mathbb{S}^1}(M, F)$ and from the exact sequence

$$\cdots \rightarrow [H(B, F) \otimes \Lambda(e)]^i \rightarrow H_{\mathbb{S}^1}^i(M, F) \rightarrow [H(B, F) \otimes \Lambda(e)]^{i-1} \xrightarrow{q^*} [H(B, F) \otimes \Lambda(e)]^{i+1} \rightarrow \cdots,$$

obtained proceeding as in d). In fact, for $S = \{1, e^p, e^{2p}, \dots\}$, the inverse of ∇ is given by

$$\nabla^{-1}([\omega]) = \sum_{n \geq 0} \sum_{j=0}^{p-1} (-1)^{(n+1)p-1-j} (e^{-p})^{n+1} e^j \cdot [\omega \wedge e^{p(n+1)-j-1}],$$

which makes sense since the differential form e^m vanishes for large enough m .

4.8 Semifree \mathbb{S}^3 -actions. The results developed until here for circle actions extend directly to semifree \mathbb{S}^3 -actions. This comes essentially from the fact that formula (6) applies (up to a shift) here (cf. [19]). We don't give all the results but just the main one.

Consider $\Phi: \mathbb{S}^3 \times M \rightarrow M$ a semifree smooth action of \mathbb{S}^3 on a smooth manifold M . Write F the submanifold of fixed points. Semifreeness means that \mathbb{S}^3 acts freely on $M - F$. The orbit space B is a stratified pseudomanifold whose singular strata are the connected components of F . The inclusion $\iota: F \hookrightarrow B$ induces the inclusion operator $i: \Omega_v(B, F) \rightarrow \Omega_v(B)$ which is an $\mathcal{A}(B)$ -dgm morphism of degree 0. Here the Euler form e lies on $\Omega^4(B - F)$ and it induces the $\mathcal{A}(B)$ -dgm morphism $e: \Omega_v(B, F) \rightarrow \Omega_v(B)$ of degree 4. The $\mathcal{A}(B)$ -dgm minimal models of these operators are i' and e' respectively. The main result in this framework is

Theorem 4.8.1

$$\begin{aligned}\mathcal{M}(M) &= \mathcal{A}(B) \oplus_{e'} \mathcal{M}(B, F) \\ \mathcal{M}(F) &= \mathcal{A}(B) \oplus_{i'} \mathcal{M}(B, F)\end{aligned}$$

Proof. Follow the path taken in the Theorem 4.3.1. ♣

4.9 Isometric flows. An *isometric flow* is a real smooth action $\Phi: \mathbb{R} \times M \rightarrow M$ preserving a riemannian metric μ on the smooth manifold M . The fundamental vector field X of the action is a Killing vector field. We shall write \mathcal{F} the singular foliation determined by the orbits of the action. The fixed point set is a manifold written F . Notice that in this case the orbit space $B = M/\mathbb{R}$ can be very wild (even totally disconnected!). For this reason it is customary to work with “basic objects” (objects living on M transverse to the flow and invariants by the flow) instead to work directly with the objects living on B . For example, a *basic form* is a differential form on M which is transverse to the flow ($i_X \omega = 0$) and invariant by the flow ($L_X \omega = 0$), a *basic controlled form* is a controlled form on M verifying $i_X \omega = i_X d\omega = 0$, ... We shall write

$$\begin{aligned}\Omega(M/\mathcal{F}) &\quad \text{the complex of basic forms} \\ \Omega_v(M/\mathcal{F}) &\quad \text{the complex of basic controlled forms} \\ \Omega_v((M, F)/\mathcal{F}) &\quad \text{the complex of basic relative controlled forms.}\end{aligned}$$

When the action is periodic we have in fact a circle action and these complexes become, up to isomorphism, $\Omega(B)$, $\Omega_v(B)$ and $\Omega_v(B, F)$ respectively.

The three complexes above are in fact dgc algebras. The dgca minimal model of $\Omega_v(M/\mathcal{F})$ and $\Omega(M/\mathcal{F})$ are the same and they will be denoted by $\mathcal{A}(M/\mathcal{F})$ (cf. [20]). We work in the category of $\mathcal{A}(M/\mathcal{F})$ -dg modules. The $\mathcal{A}(M/\mathcal{F})$ -dgm minimal model of $\Omega_v((M, F)/\mathcal{F})$ will be denoted by $\mathcal{M}((M, F)/\mathcal{F})$.

The inclusion $\iota: F \hookrightarrow M$ induces the inclusion operator $i: \Omega_v((M, F)/\mathcal{F}) \rightarrow \Omega_v(M/\mathcal{F})$ which is an $\mathcal{A}(M/\mathcal{F})$ -dgm morphism of degree 0. Here the Euler form e lies on $\Omega^2((M - F)/\mathcal{F})$ and it induces the $\mathcal{A}(M/\mathcal{F})$ -dgm morphism $e: \Omega_v((M, F)/\mathcal{F}) \rightarrow \Omega_v(M/\mathcal{F})$ of degree 2. The $\mathcal{A}(M/\mathcal{F})$ -dgm minimal models of these operators are i' and e' respectively.

We shall write $X = \partial/\partial t$ if there exists a diffeomorphism $M \cong B \times \mathbb{R}$ sending X on a multiple of $\partial/\partial t$. The main result in this framework is

Theorem 4.9.1 *If $X \neq \partial/\partial t$ then*

$$\begin{aligned}\mathcal{M}(M) &= \mathcal{A}(M/\mathcal{F}) \oplus_{e'} \mathcal{M}((M, F)/\mathcal{F}) \\ \mathcal{M}(F) &= \mathcal{A}(M/\mathcal{F}) \oplus_{i'} \mathcal{M}((M, F)/\mathcal{F})\end{aligned}$$

Proof. If we prove that the inclusion $I\Omega_v(M) \hookrightarrow \Omega_v(M)$ and the restriction $\Omega(M) \rightarrow \Omega_v(M)$ are dgca quis, then it suffices to follow the path taken in the proof of the Theorem 4.3.1.

An isometric flow defines a singular riemannian foliation \mathcal{F} where the closures of the orbits are all closed or all tori (cf. [11]). In the first case the natural projection $\pi: M \rightarrow M/\mathcal{F}$ becomes a locally trivial fibration and, by orientability, a trivial one. So, $X = \partial/\partial t$ and we are in the second case. Using the Mayer-Vietoris argument we can replace M by a torus \mathbb{T} endowed with an \mathbb{R} -linear action. The problem becomes to prove that the inclusion $I\Omega_v(\mathbb{T}) \hookrightarrow \Omega_v(\mathbb{T})$ and the restriction $\Omega(\mathbb{T}) \rightarrow \Omega_v(\mathbb{T})$ are dgca quis. Since the flow is regular then $\Omega_v(\mathbb{T}) = \Omega(\mathbb{T})$. By density, the complex $I\Omega_v(\mathbb{T}) = I\Omega(\mathbb{T})$ becomes $\Omega_{\mathbb{T}}(\mathbb{T}) = \{\omega \in \Omega(\mathbb{T}) \text{ invariant by } \mathbb{T}\}$. Since \mathbb{T} is compact, we already know that the inclusion $\Omega_{\mathbb{T}}(\mathbb{T}) \hookrightarrow \Omega(\mathbb{T})$ is a dgca quis (cf. [5]). \clubsuit

In the case $X = \partial/\partial t$ this result would imply $H^*(M) = H^*(B \times \mathbb{S}^1)$, which is false. We finish the work extending to isometric flows a well known result related to periodic actions.

Corollary 4.9.2 *If $X \neq \partial/\partial t$ then, for any $r \geq 0$:*

$$H^{r-1}((M, F)/\mathcal{F}) + \sum_{i=0}^{\infty} \dim H^{r+2i}(F) \leq \sum_{i=0}^{\infty} \dim H^{r+2i}(M).$$

Proof. From the above formulae we get the following long exact sequences:

$$\begin{aligned} \cdots \rightarrow H^i(M/\mathcal{F}) \rightarrow H^i(M) \rightarrow H^{i-1}((M, F)/\mathcal{F}) \rightarrow H^{i+1}(M/\mathcal{F}) \rightarrow \cdots, \\ \cdots \rightarrow H^i((M, F)/\mathcal{F}) \rightarrow H^i(M/\mathcal{F}) \rightarrow H^i(F) \rightarrow H^{i+1}((M, F)/\mathcal{F}) \rightarrow \cdots. \end{aligned}$$

Since the action is free out of F the above Theorem admits the relative version: $\mathcal{M}(M, F) = \mathcal{M}((M, F)/\mathcal{F}) \oplus_{\epsilon'} \mathcal{M}((M, F)/\mathcal{F})$. This gives the long exact sequence

$$\cdots \rightarrow H^i((M, F)/\mathcal{F}) \rightarrow H^i(M, F) \rightarrow H^{i-1}((M, F)/\mathcal{F}) \rightarrow H^{i+1}((M, F)/\mathcal{F}) \rightarrow \cdots.$$

Finally, by considering the long exact sequence associated to (M, F) one gets all the ingredients to proceed as in [3, pag.161] in order to obtain the following Smith-Gysin sequence:

$$\cdots \rightarrow H^i((M, F)/\mathcal{F}) \rightarrow H^i(M) \rightarrow H^{i-1}((M, F)/\mathcal{F}) \oplus H^i(F) \rightarrow H^{i+1}((M, F)/\mathcal{F}) \rightarrow \cdots$$

Now it suffices to follow the procedure of [3, pag. 127]. \clubsuit

When M is compact the group of isometries of (M, μ) is a compact Lie group. So, the action of \mathbb{R} extends to an action of a torus \mathbb{T} . For this action we have the inequality

$$H^{r-1}((M, F)/\mathbb{T}) + \sum_{i=0}^{\infty} \dim H^{r+2i}(F) \leq \sum_{i=0}^{\infty} \dim H^{r+2i}(M),$$

for each $r \geq 0$ (cf. [3]), but notice that in general $H^*((M, F)/\mathcal{F})$ and $H^*((M, F)/\mathbb{T})$ are not equal.

5 Appendix

In this Appendix we give the proof of Theorem 1.3.1 (existence of a minimal KS-extension), Theorem 1.3.3 (uniqueness of a minimal KS-extension), Theorem 3.3.1 (minimal model versus controlled forms) and Theorem 3.4.1 (relative minimal model versus controlled forms).

5.1 Proof of Theorem 1.3.1.

To begin with, put $N(0, 0) = M$, $\iota_{(0,0)} = \text{id}_M$ and $\rho_{(0,0)} = \varphi$. Let us assume that we have already constructed

$$\iota_{(n,0)}: M \longrightarrow N(n, 0) \quad \text{and} \quad \rho_{(n,0)}: N(n, 0) \longrightarrow X$$

such that

- (i)_(n,0) $\iota_{(n,0)}$ is a minimal KS-extension,
- (ii)_(n,0) $\rho_{(n,0)}\iota_{(n,p)} = \varphi$, and
- (iii)_(n,0) $(\rho_{(n,0)})_*^i: H^i(N(n, 0)) \rightarrow H^i(X)$ is an isomorphism for $0 \leq i \leq n-1$ and $(\rho_{(n,0)})_*^n$ is a monomorphism.

Now, for $q > 0$, take $V(n, q) = H^{n+1}(N(n, q-1), X)$ and consider it as a homogeneous vector space of degree n . Take also a linear section s of the natural projection $Z^{n+1}(N(n, q-1), X) \rightarrow V(n, q)$. So for every $v \in V(n, q)$, we have $sv = (t_v, x_v) \in N(n, q-1)^{n+1} \oplus X^n$ such that

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} d & 0 \\ \rho_{(n,q-1)} & -d \end{pmatrix} \begin{pmatrix} t_v \\ x_v \end{pmatrix}.$$

Define

$$N(n, q) = N(n, q-1) \oplus_d (A \otimes V(n, q))$$

with differential $dv = t_v$. Define also $\iota_{(n,q)}: M \rightarrow N(n, q)$ as the composition of $\iota_{(n,q-1)}$ with the natural inclusion of $N(n, q-1) \hookrightarrow N(n, q)$. Finally, take $\rho_{(n,q)}: N(n, q) \rightarrow X$ to be the morphism of A -dg modules induced by $\rho_{(n,q-1)}$ and the linear map $f: V(n, q) \rightarrow X$ defined by $fv = x_v$.

Let us verify that:

- (i)_(n,q) $\iota_{(n,q)}$ is a minimal KS-extension because of (i)_(n,q-1) and the definition of $\iota_{(n,q)}$,
- (ii)_(n,q) $\rho_{(n,q)}\iota_{(n,q)} = \rho_{(n,q-1)}\iota_{(n,q-1)} = \varphi$ because of the definitions and (ii)_(n,q-1),
- (iii)_(n,q) $(\rho_{(n,q)})_*^i: H^i(N(n, q)) \rightarrow H^i(X)$ is an isomorphism for $i = 0, \dots, n$.

In fact, for $i < n$, $\rho_{(n,q)}$ coincides with $\rho_{(n,q-1)}$ and so the morphism induced in cohomology is an isomorphism by (iii)_(n,q-1). In degree n , the natural inclusion $N(n, q-1) \hookrightarrow N(n, q)$ induces a monomorphism in cohomology because new generators can only kill cocycles in degree $> n$. So $(\rho_{(n,q)})_*^n$ is a monomorphism and the only thing we have to prove is that it is also an epimorphism: let $[x] \in H^n(X)$ and $[x] \notin (\rho_{(n,q-1)})_*^n$ (otherwise we are done). Then x defines a non zero relative cohomology class $v = [(0, x)] \in V(n, q)$. By definition, $v = [(dv, \rho_{(n,q)})v]$. So

$$\begin{pmatrix} dv \\ \rho_{(n,q)}v - x \end{pmatrix} = \begin{pmatrix} d & 0 \\ \rho_{(n,q-1)} & -d \end{pmatrix} \begin{pmatrix} t \\ y \end{pmatrix} = \begin{pmatrix} dt \\ \rho_{(n,q-1)}t - dy \end{pmatrix}$$

with $t \in N(n, q-1)^{n+1}$, $y \in X^n$. As a consequence,

$$(\rho_{(n,q)})_*^n[v - t] = [x]$$

and so $(\rho_{(n,q)})_*^n$ is an epimorphism. Lastly, put $N(n+1,0) = \varinjlim_q N(n,q)$, $\iota_{(n+1,0)} = \varinjlim_q \iota_{(n,q)}$, $\rho_{(n+1,0)} = \varinjlim_q \rho_{(n,q)}$ and let us verify the induction hypothesis. Conditions $(i)_{(n+1,0)}$ and $(ii)_{(n+1,0)}$ are trivial. For $(iii)_{(n+1,0)}$, $(\rho_{(n+1,0)})_*^i: H^i(N(n+1,0)) \rightarrow H^i(X)$ is an isomorphism for $i = 0, \dots, n$ because all of the $(\rho_{(n,q)})_*^i$ are isomorphisms by $(iii)_{(n,q)}$ and $(\rho_{(n+1,0)})_*^{n+1}$ is a monomorphism, because if $a \in Z^{n+1}N(n+1,0)$ is such that $[\rho_{(n+1,0)}a] = 0$, then, as $a \in N(n,q)$ for some q , we would have $\rho_{(n,q)}a = dx$ for some $x \in X$. So $v = [(a, x)] \in V(n, q+1)$ will kill the class $[a]$ in $H^{n+1}(N(n,q))$ and therefore in $H^{n+1}(N(n+1,0))$. ♣

5.2 Proof of Theorem 1.3.3. We will confine ourselves to the case $M = 0$, since this is the only case we need in this paper. Let $N = A \otimes V$, and assume we have already built

$$\sigma_{(m,0)}: N(m,0) \rightarrow X$$

in a such way that for all $m \leq n$

$$(i)_m \quad \rho\sigma_{(m,0)} = \text{id}_{N(m,0)}, \text{ and}$$

$$(ii)_m \quad \sigma_{(m,0)}|_{N(m',0)} = \sigma_{(m',0)} \text{ for all } m' \leq m.$$

We will define

$$\sigma_{(n+1,0)}: N(n+1,0) \rightarrow X$$

satisfying $(i)_{n+1}$, $(ii)_{n+1}$ and then the section of ρ will be

$$\sigma = \varinjlim_n \sigma_{(n,0)}.$$

To do this, we will extend $\sigma_{(n,0)}$ to morphisms

$$\sigma_{(n,q)}: N(n,q) \rightarrow X$$

in such a way that for every $p \leq q$

$$(i)_{(n,p)} \quad \rho\sigma_{(n,p)} = \text{id}_{N(n,p)}, \text{ and}$$

$$(ii)_{(n,p)} \quad \sigma_{(n,p)}|_{N(n,p')} = \sigma_{(n,p')}, \text{ for all } p' \leq p.$$

Once we have these $\sigma_{(n,q)}$ for all $q \geq 0$, we will put

$$\sigma_{(n+1,0)} = \varinjlim_q \sigma_{(n,q)},$$

which will verify $(i)_{n+1}$ and $(ii)_{n+1}$.

So, let us begin the construction of the $\sigma_{(n,q)}$. For $q = 0$, $\sigma_{(n,0)}$ exists by induction hypothesis. Assume we already have $\sigma_{(n,p)}$ for all $p \leq q$. Then, consider the A -dg module

$$X(n,q) = \text{Im} \left(\sigma_{(n,q)}: N(n,q) \rightarrow X \right).$$

Because $\rho\sigma_{(n,q)} = \text{id}_{N(n,q)}$, $\sigma_{(n,q)}$ is a monomorphism and so $\sigma_{(n,q)}: N(n,q) \rightarrow X(n,q)$ is an isomorphism. If we apply the Five Lemma to the long cohomology sequences of the following commutative diagram of exact sequences of A -dg modules

$$\begin{array}{ccccccc}
 0 & \longrightarrow & X(n, q) & \longrightarrow & X & \longrightarrow & X/X(n, q) \longrightarrow 0 \\
 & & \downarrow \sigma_{(n, q)}^{-1} & & \downarrow \rho & & \downarrow \bar{\rho} \\
 0 & \longrightarrow & N(n, q) & \longrightarrow & N & \longrightarrow & N/N(n, q) \longrightarrow 0
 \end{array}$$

we obtain that $\bar{\rho}$ is a quies of A -dg modules.

Next, let $V(n, q+1)$ be an \mathbb{R} -vector space such that

$$N(n, q+1) = N(n, q) \oplus_d (A \otimes V(n, q+1))$$

and let j be the composition $V \xrightarrow{i} N \rightarrow N/N(n, q)$. Since $dV \subset N(n, q)$, we have $jV \subset Z(N/N(n, q))$, and so we have an \mathbb{R} -linear morphism $H_j: V \rightarrow H(N/N(n, q))$ which we can lift to $Z(X/X(n, q))$:

$$\begin{array}{ccccc}
 & & & & Z(X/X(n, q)) \\
 & & & \nearrow \lambda & \downarrow p \\
 V & \xrightarrow{H_j} & H(N/N(n, q)) & \xleftarrow{\cong} & H(X/X(n, q))
 \end{array}$$

In fact, $N/N(n, q)$ is an A -dg module concentrated in degrees $\geq n$. So $B^n(N/N(n, q)) = 0$ and the previous diagram in degree n is just

$$\begin{array}{ccc}
 & & Z^n(X/X(n, q)) \\
 & \nearrow \lambda & \downarrow \bar{\rho} \\
 V & \xrightarrow{j} & Z^n(N/N(n, q))
 \end{array}$$

Next, consider the pull back

$$\begin{array}{ccc}
 (X/X(n, q)) \times_{N/N(n, q)} N & \longrightarrow & N \\
 \downarrow & & \downarrow \\
 X/X(n, q) & \xrightarrow{\bar{\rho}} & N/N(n, q)
 \end{array}$$

and the morphism induced by $\pi: X \rightarrow X/X(n, q)$ and $\rho: X \rightarrow N$,

$$(\pi, \rho): X \longrightarrow (X/X(n, q)) \times_{N/N(n, q)} N.$$

It is an epimorphism: if $(\tilde{x}, y) \in (X/X(n, q)) \times_{N/N(n, q)} N$ then $\rho x - y \in N(n, q)$. So,

$$\begin{aligned}
 (\pi, \rho)(x - \sigma_{(n, q)}(\rho x - y)) &= (\tilde{x} - 0, \rho x - (\rho x - y)) \\
 &= (\tilde{x}, y).
 \end{aligned}$$

So, we can lift (λ, i) to X

$$\begin{array}{ccc}
 & & X \\
 & \nearrow f & \downarrow (\pi, \rho) \\
 V & \xrightarrow{(\lambda, i)} & (X/X(n, q)) \times_{N/N(n, q)} N
 \end{array}$$

Then f and $\sigma_{(n, q)}$ induce the morphism of A -dg modules $\sigma_{(n, q+1)}: N(n, q+1) \rightarrow X$ we were looking for:

$$\sigma_{(n, q+1)}|_{N(n, q)} = \sigma_{(n, q)} \text{ and } \sigma_{(n, q+1)}|_V = f.$$

To see it is an A -dg morphism, it suffices, by definition of a Hirsch extension, to verify that

$$\sigma_{(n, q)}d = df.$$

So, let $v \in V$, then $\pi f v = \lambda v \in Z^n(X/X(n, q))$. So, $0 = d\pi f v = \pi df v$. Hence $df v \in X(n, q) = \text{Im } \sigma_{(n, q)}$. Let $\omega \in N(n, q)$ be such that $df v = \sigma_{(n, q)}\omega$. Apply ρ to both sides of this equality and get, by the previous diagram,

$$\rho df v = d\rho f v = dv$$

and, by induction hypothesis $(i)_{(n, q)}$,

$$\rho \sigma_{(n, q)}\omega = \omega.$$

So $dv = \omega$, as we wanted.

Finally, $\sigma_{(n, q+1)}$ verifies $(i)_{(n, q+1)}$ and $(ii)_{(n, q+1)}$ by construction. ♣

5.3 Proof of Theorem 3.3.1.

This result generalizes to stratified spaces the result asserting that the dgca minimal model of a manifold can be computed using its deRham complex. The proof is adapted from [8] whose notation is used. It will be sufficient to prove that the dgca minimal model of $\Omega_v(Z)$ is that of $A_{\mathbb{R}}(Z)$. For this purpose we construct a commutative diagram

$$\begin{array}{ccccc}
 \Omega_v(Z) & \xrightarrow{\aleph_1} & E(\underline{LS}(Z)) & \xleftarrow{\aleph_2} & A_{\mathbb{R}}(Z) \\
 \downarrow f_2 & & \nearrow f_3 & & \downarrow f_1 \\
 C(\underline{LS}(Z)) & & & \xleftarrow{\aleph_3} & C(\underline{Sing}(Z))
 \end{array}
 \tag{9}$$

where f_1, f_2, f_3 , and \aleph_3 induce cohomology isomorphisms and \aleph_2 et \aleph_1 are dgca morphisms. This implies that \aleph_2 and \aleph_1 are also dgca quasi-isomorphisms and the Proposition is proved. We construct the diagram in several steps.

I. Unfolding of Δ (see [18]). The *unfolding* of the standard simplex Δ , relatively to the decomposition $\Delta = \Delta_0 * \dots * \Delta_p$, is the map

$\mu_{\Delta}: \tilde{\Delta} = \bar{c}\Delta_0 \times \dots \times \bar{c}\Delta_{p-1} \times \Delta_p \longrightarrow \Delta$ defined by $\mu_{\Delta}([x_0, t_0], \dots, [x_{p-1}, t_{p-1}], x_p) = t_0 x_0 + (1 - t_0)t_1 x_1 + \dots + (1 - t_0) \dots (1 - t_{p-2})t_{p-1}x_{p-1} + (1 - t_0) \dots (1 - t_{p-1})x_p$. Here $\bar{c}\Delta_i$ denotes the closed cone $\Delta_i \times [0, 1]/\Delta_i \times \{0\}$ and $[x_i, t_i]$ a point of it. This map is smooth and its restriction $\mu_{\Delta}: \text{int}(\tilde{\Delta}) \longrightarrow \text{int}(\Delta)$ is a diffeomorphism (we write $\text{int}(P) = P - \partial P$ the *interior* of the polyhedron P). It sends a face U of $\tilde{\Delta}$ on a face V of Δ and the restriction $\mu_{\Delta}: \text{int}(U) \rightarrow \text{int}(V)$ is a submersion.

This blow up is compatible with face and degeneracy maps.

1. *Face.* Put $\delta_F: F \rightarrow \Delta$ a codimension one face of Δ , the induced decomposition is $F = \Delta_0 * \cdots * \Delta_{j-1} * F_j * \Delta_{j+1} * \cdots * \Delta_p$ (we have written $\emptyset * X = X$). The *lifting* of δ_F is the map $\tilde{\delta}_F: \tilde{F} \rightarrow \tilde{\Delta}$ defined by:

$$\tilde{\delta}_F(z) = \begin{cases} z & \text{if } F_j \neq \emptyset \\ (z_0, \dots, z_{j-1}, \vartheta_j, z_{j+1}, \dots, z_{p-1}, x_p) & \text{if } F_j = \emptyset, j \neq p, \vartheta_j \text{ is the vertex of } \bar{c}\Delta_j \text{ and} \\ & z = (z_0, \dots, \hat{z}_j, \dots, z_{p-1}, x_p) \\ (z_0, \dots, z_{p-2}, [x_{p-1}, 1]) & \text{if } F_j = \emptyset, j = p, \text{ and } z = (z_0, \dots, z_{p-2}, x_{p-1}). \end{cases}$$

This map is smooth (affine in barycentric coordinates), sends isomorphically \tilde{F} on a face of $\tilde{\Delta}$ and verifies $\mu_\Delta \circ \tilde{\delta}_F = \delta_F \circ \mu_F$.

2. *Degeneracy.* Put $\sigma: D = \Delta * \{P\} \rightarrow \Delta$ a degeneracy map with $\sigma_D(P) = Q \in \Delta_j$. The induced decomposition $D = \Delta_0 * \cdots * \Delta_{j-1} * (\Delta_j * \{P\}) * \Delta_{j+1} * \cdots * \Delta_p$. The *lifting* of σ is the map $\tilde{\sigma}_D: \tilde{D} \rightarrow \tilde{\Delta}$ defined by:

$$\tilde{\sigma}_D(z) = \begin{cases} (z_0, \dots, [tx_j + (1-t)Q, t_j], \dots, z_{p-1}, x_p) & \text{if } j < p \text{ and } z \text{ is} \\ & (z_0, \dots, [tx_j + (1-t)P, t_j], \dots, z_{p-1}, x_p), \\ (z_0, \dots, z_{p-1}, tx_p + (1-t)Q) & \text{if } j = p \text{ and } z \text{ is} \\ & (z_0, \dots, z_{p-1}, tx_p + (1-t)P). \end{cases}$$

This map is smooth (linear in barycentric coordinates) and verifies $\mu_\Delta \circ \tilde{\sigma}_D = \sigma_D \circ \mu_D$.

On the boundary $\partial\tilde{\Delta}$ we find not only the blow up $\tilde{\partial}\Delta$ of the boundary $\partial\Delta$ of Δ but also the faces $B_i = \bar{c}\Delta_0 \times \cdots \times \bar{c}\Delta_{i-1} \times (\Delta_i \times \{1\}) \times \bar{c}\Delta_{i+1} \times \cdots \times \bar{c}\Delta_{p-1} \times \Delta_p$ with $i \in \{0, \dots, p-2\}$ or $i = p-1$ and $\dim \Delta_p > 0$, that we shall call *bad faces*. Notice that $\dim \mu_\Delta(B_i) = \dim(\Delta_0 * \cdots * \Delta_i) < n - 1 = \dim B_i$.

II. The simplicial set²

LS(Z). On a stratified pseudomanifold it is not possible to define directly smooth simplices as in [8]. For this reason we introduce the notion of liftable simplices. Put $Z = Z_{n=\dim R} \supset \Sigma_Z = Z_{n-1} \supset \cdots \supset Z_0 \supset Z_{-1} = \emptyset$ the filtration of Z , that is, Z_i is the union of strata with dimension smaller than i . A *liftable simplex* is a singular simplex $\varphi: \Delta \rightarrow Z$ verifying the two following conditions.

[LS1] *Each pull back $\varphi^{-1}(Z_i)$ is a face of Δ .*

Consider $\{i_0, \dots, i_p\} = \{i \in \{0, \dots, n\} / \varphi^{-1}(Z_i) \neq \varphi^{-1}(Z_{i-1})\}$ and put Δ_j the face of Δ with $\varphi^{-1}(Z_{i_j}) = \varphi^{-1}(Z_{i_{j-1}}) * \Delta_j$. This defines on Δ the φ -*decomposition* $\Delta = \Delta_0 * \cdots * \Delta_p$.

[LS2] *There exists a smooth map $\tilde{\varphi}: \tilde{\Delta} \rightarrow \tilde{Z}$ with $\mathcal{L}_Z \circ \tilde{\varphi} = \varphi \circ \mu_\Delta$, where the unfolding of Δ is taken relatively to the φ -decomposition of Δ .*

The $\varphi \circ \delta_F$ -decomposition (resp. $\varphi \circ \sigma_D$ -decomposition) is just $F = \Delta_0 * \cdots * \Delta_{j-1} * F_j * \Delta_{j+1} * \cdots * \Delta_p$ (resp. $D = \Delta_0 * \cdots * \Delta_{j-1} * (\Delta_j * \{P\}) * \Delta_{j+1} * \cdots * \Delta_p$). So, for a liftable simplex φ the simplices $\partial_F(\varphi) = \varphi \circ \delta_F$ and $s_D(\varphi) = \varphi \circ \sigma_D$ are liftable simplices. The face map ∂_F and the degeneracy map s_D verify the usual compatibility conditions. Put LS(Z) the family of liftable simplices. So, (LS(Z), ∂, s) define a simplicial set.

Notice that a liftable simplex φ sends the interior of a face A of Δ on a stratum S of Z and that the restriction $\varphi: \text{int}(A) \rightarrow S$ is smooth (using face maps we can suppose $A = \Delta$ and there we know that $\mu_\Delta: \text{int}(\tilde{\Delta}) \rightarrow \text{int} \Delta$ is a diffeomorphism).

²For the notions related with simplicial sets, local systems, ... we refer the reader to [8]

III. The local systems C and E . The local system C on $\text{Sing}(Z)$ (resp. $\underline{LS}(Z)$) is defined in [8, 14.2] in such a way that the space of global sections $C(\underline{\text{Sing}}(Z))$ (resp. $C(\underline{LS}(Z))$) is the complex generated by the singular simplices (resp. liftable simplices) of Z .

Consider a simplex Δ endowed with the decomposition $\Delta_0 * \cdots * \Delta_p$. A *liftable form* on Δ is a family of differential forms $\eta = \{\eta_A \in \Omega(\text{int}(A)) \mid A \text{ face of } \Delta\}$ possessing a common lifting $\tilde{\eta} \in \Omega(\tilde{\Delta})$, that is:

$$\tilde{\eta} = \mu_{\Delta}^* \eta_{\mu_{\Delta}(H)} \text{ on } \text{int}(H), \text{ for each face } H \text{ of } \tilde{\Delta}.$$

The lifting $\tilde{\eta}$ is unique. Since μ_{Δ} is an onto submersion with connected fibers then the lifting forms are exactly the differential forms ω on $\tilde{\Delta}$ verifying $\omega(v, -) = d\omega(v, -) = 0$ for any vector of $\tilde{\Delta}$ with $(\mu_{\Delta})_*(v) = 0$, that is, the *basic forms* on $\tilde{\Delta}$. In the sequel we shall use both of points of view.

A *liftable form* on $\{\varphi: \Delta \rightarrow Z\} \in \underline{LS}(Z)$ is a liftable form on Δ relatively to its φ -decomposition. For any $\varphi: \Delta \rightarrow Z$ liftable simplex we shall write $E_{\varphi} = \{\text{liftable forms on } \varphi\}$ which is a dgca complex.

Consider $\delta_F: F \rightarrow \Delta$ a face map and $\sigma_D: D \rightarrow \Delta$ a degeneracy map. We define the face operator $\partial_F: E_{\varphi} \rightarrow E_{\partial_F(\varphi)}$ and the degeneracy operator $s_D: E_{\varphi} \rightarrow E_{s_D(\varphi)}$ by $\partial_F(\tilde{\eta}) = \tilde{\delta}_F^* \tilde{\eta}$ and $s_D(\tilde{\eta}) = \tilde{\sigma}_D^* \tilde{\eta}$ respectively. These operators verify the usual compatibility conditions and thus define a local system E on $\underline{LS}(Z)$. Notice that the space $E(\underline{LS}(Z))$ of global sections of E is a dgc algebra.

When Z is a manifold endowed with the stratification $\{Z\}$, liftable simplex becomes smooth simplex and so $\underline{LS}(Z) = \underline{\text{Sing}}^{\infty}(Z)$. Moreover, E becomes the local system A_{∞} of C^{∞} differential forms.

IV. Operators \aleph and \int .

- The operator \aleph_3 is just the inclusion, which makes sense since any liftable simplex is a singular one. Proceeding as in [18] one proves that this inclusion induces an isomorphism in homology and therefore that \aleph_3 is a quasi-isomorphism in the category of graded vector spaces.

- The operator \aleph_2 is defined as follows. For each $\omega \in A_{\mathbb{R}}(Z)$ and each liftable simplex φ of Z we put $\mu_{\Delta}^* \omega_{\varphi}$ the lifting of $(\aleph_2(\omega))_{\varphi}$. This operator is dgca operator.

- The operator \aleph_1 is defined as follows. For each $\omega \in \Omega_v(Z)$ and each liftable simplex φ of Z we put $\tilde{\varphi}^* \tilde{\omega}$ the lifting of $(\aleph_1(\omega))_{\varphi}$. This operator is dgca operator.

- The operator \int_1 is given by integration of differential forms on simplices:

$$\left(\int_1 \omega\right)(\varphi) = \int_{\Delta} \omega_{\varphi} \quad \text{where } \omega \in A_{\mathbb{R}}(Z) \text{ and } \varphi \text{ is a singular simplex of } Z.$$

The deRham Theorem says that \int_1 is a quasi-isomorphism in the category of graded vector spaces [8].

- The operator \int_2 is given by integration of differential forms on simplices:

$$\left(\int_2 \omega\right)(\varphi) = \int_{\text{int}(\Delta)} \varphi^* \omega = \int_{\tilde{\Delta}} \tilde{\varphi}^* \tilde{\omega} \quad \text{where } \omega \in \Omega_v(Z) \text{ and } \varphi \text{ is a liftable simplex of } Z.$$

This operator is differential if we have $\int_{B_i} \tilde{\varphi}^* \tilde{\omega} = 0$. To prove that we write S the stratum of Z containing $\varphi(\text{int}(\mu_{\Delta}(B_i)))$; since $\tilde{\varphi}^* \tilde{\omega} = \mu_{\Delta}^* \varphi^* \omega_S$ then we get $\int_{B_i} \tilde{\varphi}^* \tilde{\omega} = 0$ because $\dim B_i >$

$\dim \mu_\Delta(B_i)$. Proceeding as in [18] one proves that \int_2 is a quasi-isomorphism in the category of graded vector spaces.

- The operator \int_3 is given by integration of differential forms on simplices:

$$(\int_3 \eta)(\varphi) = \int_{\text{int}(\Delta)} (\eta_\varphi)_\Delta = \int_{\tilde{\Delta}} \tilde{\eta}_\varphi \quad \text{áwhere } \eta \in E(\underline{LS}(Z)) \text{ and } \varphi \text{ is a liftable simplex of } Z.$$

This operator is differential since $\int_{B_i} \tilde{\eta}_\varphi = 0$; this comes from the equality $\tilde{\eta}_\varphi = \mu_\Delta^*(\eta_\varphi)_{\mu_\Delta(B_i)}$ on $\text{int}(B_i)$. We already know that the local system C on $\underline{LS}(Z)$ is an extendable local system [8, Proposition 14.11], if we prove that D is a extendable local system we shall get that \int_3 is a quasi-isomorphism in the category of graded vector spaces [8, Theorem 12.27]. This fact comes from the Poincaré Lemma and the Extension Property we prove below. For this purpose we fix a decomposition $\Delta = \Delta_0 * \cdots * \Delta_p$.

Poincaré Lemma. $H(\{\text{liftable forms on } \Delta\}) = \mathbb{R}$.

We prove $H(\{\text{basic forms on } \tilde{\Delta}\}) = \mathbb{R}$. Suppose first the case $\dim \Delta_p > 0$. Put ϑ vertex of Δ_p . Consider the following maps:

$$\begin{aligned} h_1: \Delta \times [0, 1] &\rightarrow \Delta \text{ defined by } h_1(r_0x_0 + \cdots + r_px_p, t) = r_0x_0 + \cdots + r_{p-1}x_{p-1} + r_p(t\vartheta + (1-t)x_p), \\ h_2: \tilde{\Delta} \times [0, 1] &\rightarrow \tilde{\Delta} \text{ defined by } h_2(z_0, \dots, z_{p-1}, x_p, t) = (z_0, \dots, z_{p-1}, t\vartheta + (1-t)x_p). \end{aligned}$$

They are smooth homotopy maps between Δ (resp. $\tilde{\Delta}$) and $\Delta' = \Delta_0 * \cdots * \Delta_{p-1} * \{\vartheta\}$ (resp. $\tilde{\Delta}'$). Since $\mu_\Delta(h_2(z_0, \dots, z_{p-1}, x_p, t)) = h_1(\mu_\Delta(z_0, \dots, z_{p-1}, x_p), t)$ the basic forms are preserved and h_2 induce a homotopy operator between $\{\text{basic forms on } \tilde{\Delta}\}$ and $\{\text{basic forms on } \tilde{\Delta}'\}$.

Suppose $\dim \Delta_p = 0$, that is, $\Delta_p = \{\vartheta\}$. Write ∇ the simplex Δ endowed with the decomposition $\Delta_0 * \cdots * \Delta_{p-2} * (\Delta_{p-1} * \{\vartheta\})$. Since the map $\tau: \tilde{\nabla} \rightarrow \tilde{\Delta}$, defined by $\tau(z_0, \dots, z_{p-2}, tx_{p-1} + (1-t)\vartheta) = (z_0, \dots, z_{p-2}, [x_{p-1}, t], \vartheta)$, is a diffeomorphism verifying $\mu_\Delta \circ \tau = \mu_\nabla$ we get that the complexes $\{\text{basic forms on } \tilde{\Delta}\}$ and $\{\text{basic forms on } \tilde{\nabla}\}$ are isomorphic.

Applying alternatively these two procedures we arrive to the case $\Delta = \{\vartheta\}$ and here it is clear that the cohomology of $\{\text{basic forms on } \tilde{\Delta} \equiv \{\vartheta\}\}$ is \mathbb{R} .

Extension property. *Each liftable form on $\partial\Delta$ possesses an extension to a liftable form on Δ .*

Let $\eta = \{\eta_A \in \Omega(\text{int}(A)) \mid A \text{ face of } \partial\Delta\}$ be a liftable form on $\partial\Delta$. Put $\tilde{\eta} \in \Omega(\tilde{\partial\Delta})$ the lifting, which is a basic form verifying $\tilde{\eta} = \mu_\Delta^* \eta_{\mu_\Delta(H)}$ on $\text{int}(H)$ for each face H of $\tilde{\Delta}$. Since the fibers of $\mu_\Delta: \tilde{\partial\Delta} \rightarrow \partial\Delta$ are not necessarily connected then we can not identify liftable forms with basic forms.

Notice first that any form on $\partial\tilde{\Delta}$ possesses an extension to a form on $\tilde{\Delta}$. Moreover, if the form is basic then its extension is necessarily basic (the restriction $\mu_\Delta: \text{int}(\tilde{\Delta}) \rightarrow \text{int}(\Delta)$ is a diffeomorphism). So, it suffices to extend $\tilde{\eta}$ to a basic form defined on $\partial\tilde{\Delta}$.

Consider $\delta_F: F \rightarrow \Delta$ a face of $\partial\Delta$. Using face maps one constructs a smooth map $\tilde{\delta}_F: \tilde{F} \rightarrow \tilde{\Delta}$ sending isomorphically \tilde{F} on a face of $\partial\tilde{\Delta}$ and verifying $\mu_\Delta \circ \tilde{\delta}_F = \delta_F \circ \mu_F$. Notice that in this case the map $\tilde{\delta}_F$ is not necessarily unique (codim Δ F can be greater than 2). This equality implies that $\eta|_F = \{\eta_A / A \text{ face of } F\}$, restriction of η to F , is a liftable form with lifting $\tilde{\delta}_F^* \tilde{\eta}$. Recall that this form is μ_F -basic.

For each $i \in \{0, \dots, p-1\}$ we put $\nabla_i = \mu_\Delta(B_i)$, whose induced decomposition is just $\nabla_i = \Delta_0 * \dots * \Delta_i$. Define the projection $pr_i: B_i \rightarrow \tilde{\nabla}_i$ by $pr_i(z_0, \dots, z_{i-1}, (x_i, 1), z_{i+1}, \dots, z_{p-1}, x_p) = (z_0, \dots, z_{i-1}, x_i)$. This map sends (the interior of) a face of B_i on (the interior of) a face of ∇_i . We define on B_i the form $\gamma_i = pr_i^* \tilde{\delta}_{\nabla_i}^* \tilde{\eta}$, which is a basic form because $\mu_\Delta = \mu_{\nabla_i} \circ pr_i$. Moreover, γ_i is the lifting of $\eta|_{\nabla_i}$: if H is a face of B_i then $\gamma_i = pr_i^* \tilde{\delta}_{\nabla_i}^* \mu_\Delta^* \eta_{\mu_\Delta(H)} = \mu_\Delta^* \eta_{\mu_\Delta(H)}$ on $\text{int}(H)$. Since the lifting is unique the forms $\{\gamma_0, \dots, \gamma_{p-1}\}$ define a basic form on $\partial\tilde{\Delta} - \partial\Delta$, the union of bad faces. Again, the uniqueness of the lifting implies that this form coincides with $\tilde{\eta}$ on $\partial\tilde{\Delta}$. Therefore, the extension of $\tilde{\eta}$ is constructed.

• *Commutativity.* One easily checks $\int_3 \circ \aleph_1 = \int_2$ and $\int_3 \circ \aleph_2 = \aleph_3 \circ \int_1$, that is, the diagram (5) is commutative. Since $\int_1, \int_2, \int_3, \aleph_3$ induce isomorphisms in cohomology (as graded vector spaces) and \aleph_2, \aleph_1 are dgca morphisms then \aleph_2, \aleph_1 are dgca quasi-isomorphisms. So, the dgca minimal model of Z is that of $\Omega_v(Z)$. ♣

5.4 Proof of Theorem 3.4.1.

Consider the following diagram

$$\begin{array}{ccccc} \Omega_v(Z') & \xrightarrow{\aleph'_1} & E(\underline{LS}(Z')) & \xleftarrow{\aleph'_2} & A_{\mathbb{R}}(Z') \\ \uparrow f^* & & \uparrow f^* & & \uparrow f^* \\ \Omega_v(Z) & \xrightarrow{\aleph_1} & E(\underline{LS}(Z)) & \xleftarrow{\aleph_2} & A_{\mathbb{R}}(Z) \end{array}$$

where the pull backs are defined as follows.

For each $\omega \in A_{\mathbb{R}}(Z)$ and each liftable simple φ of Z' we put $(f^*\omega)_{\varphi'} = \omega_{f \circ \varphi}$. This operator is a dgca morphism.

Since f is good then $f^*: \Omega_v(Z) \rightarrow \Omega_v(Z')$ is a well defined dgca operator. For each $\eta \in E(\underline{LS}(Z))$ and each liftable simple φ of Z' we put $(f^*\eta)_{\varphi} = \eta_{f \circ \varphi}$, which makes sense since f is good. One easily checks $f^* \circ \aleph'_1 = \aleph_1 \circ f^*$ and $f^* \circ \aleph_2 = \aleph'_2 \circ f^*$ which ends the proof ♣

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